Fundamental group and contractible closed geodesics

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Abstract

We prove the existence of a non-empty class of finitely presented groups with the following property: If the fundamental group of a compact Riemannian manifold M belongs to this class then there exists a constant $c(M) > 1$ such that for any sufficiently large x the number of contractible closed geodesics on M of length not exceeding x is greater than $c(M)^x$.

In order to prove this result we give a lower bound for the number of contractible closed geodesics of length $\leqq x$ on a compact Riemannian manifold M in terms of the resource-bounded Kolmogorov complexity of the word problem for $\pi_1(M)$, thus answering a question posed by Gromov.

1. Main result

In [10] Gromov noted that if the fundamental group of a compact Riemannian manifold has an unsolvable word problem, then the manifold has infinitely many contractible closed geodesics. His paper does not contain a proof of this result but it is not difficult to prove this result by contradiction using, for example, the following idea: Assume that the set of contractible closed geodesics on M is finite. This implies the existence of the following algorithm deciding whether or not a given word w is trivial in $\pi_1(M)$ (thus, giving a contradiction to the assumption that the word problem is algorithmically unsolvable): Realize the word w by a closed curve γ_w on M. Let L be an integer number such that any contractible closed geodesic on M can be contracted to a point by a homotopy passing via closed curves of length not exceeding L. This number can be considered as a constant known to the algorithm (albeit unknown to us). Note that any contractible closed curve of length not exceeding x can be contracted to a point by a homotopy passing only via closed curves of length not exceeding $\max\{L,\ x\}$. Now the precompactness of the space of piecewise C^1-smooth curves on M of length bounded by any fixed constant and parametrized by the arc length implies the existence of a trial-and-error algorithm checking whether or not γ_w is contractible. Such an algorithm is described in details in [19] for the case when M is a non-singular algebraic hypersurface in an Euclidean space presented as the zero set of a polynomial with algebraic coefficients. The case of a variety that is non-singular algebraic over the field of real algebraic numbers and which has an arbitrary codimension in a Euclidean space is quite similar. (A description of M by a vector of coefficients of a system of defining polynomials can be regarded as a set of constants used by the algorithm.) In this case the outlined idea of the proof works without any hindrances, and one can prove even the existence of infinitely many closed geodesics on M which are "deep" local

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minima of the length functional. To deal with an arbitrary compact Riemannian manifold M consider a smooth algebraic submanifold A of $\mathbb{R}^N$ (for a sufficiently large N) C^2-close to an isometrically embedded in $\mathbb{R}^N$ copy of M. Well-known Nash and Tognoli theorems (cf. [12] and [16]) imply the existence of such a smooth algebraic submanifold of $\mathbb{R}^N$. Moreover, the Tarski-Seidenberg theorem (cf. [3]) implies that this algebraic submanifold can be defined as the zero set of a system of polynomials with algebraic coefficients (cf. [6]). Now note that the argument above implies the existence of infinitely many "deep" local minima of the length functional on the space of contractible closed curves on A, whence the existence of infinitely many contractible closed geodesics on M easily follows.

In [11] , Section 5.C, Gromov asked how one can estimate the number of contractible closed geodesics of length $\leqq l$ on a compact Riemannian manifold in terms of its fundamental group. Proposition 2, stated in section 2 and proven in section 3 below, provides an answer for this question. To obtain a quantitaive version of the argument above we consider the resource-bounded Kolmogorov complexity of the word problem for the fundamental group of the Riemannian manifold instead of mere algorithmic solvability/unsolvability of the word problem. (The notion of resource-bounded Kolmogorov complexity and its basic properties are reviewed at the beginning of section 2. In particular, we will need the Barzdin theorem ([2] , [22] , Theorem 2.5) which can be regarded as a quantitative version of the algorithmic unsolvability of the halting problem for Turing machines. Recall that, informally speaking, a resource-bounded Kolmogorov complexity of a decision problem is the minimal number of bits of oracle information required to solve the decision problem for all instances of length $\leqq l$ using amount of time and/or space bounded by prescribed function(s) of l and regarded as a function of l.) Proposition 2 provides a lower bound for the number of contractible closed geodesics of length not exceeding a given constant on a given compact Riemannian manifold M in terms of time-bounded Kolmogorov complexity of the word problem for $\pi_1(M)$. The following theorem is almost a direct corollary of Proposition 2 and the theorem of Barzdin mentioned above. It will be proven in section 2. To state it we need the following definition. An increasing function $\beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called *effectively majorizable* if there exists a Turing computable function $\alpha : \mathbb{N} \longrightarrow \mathbb{N}$ such that for any integer n $\beta(n) \leqq \alpha(n)$. (For example, the function $\beta(x) = \exp(\dots(\exp(x)))$ $([x]+1)$ times is effectively majorizable.)

THEOREM 1. *There exists a non-empty class of finitely presented groups Σ with the following property: If M is a compact Riemannian manifold such that $\pi_1(M) \in \Sigma$ then there exists a constant $c(M) > 1$ such that for any sufficiently large l the number $N(l)$ of non-constant contractible closed geodesics of length $\leqq l$ on M satisfies the inequality $N(l) > c(M)^l$. Moreover, for any increasing effectively majorizable function β there exists a number $l(M, \beta)$ such that for any $L \geqq l(M, \beta)$ there exist at least $[c(M)^L]$ contractible closed geodesics $\gamma_1, \dots, \gamma_{[c(M)^L]}$ of length $\leqq L$ with the following property: For any i, γ_i cannot* *be connected by a homotopy passing through curves of length* $\leqq \beta(length(\gamma_i))$ *neither with a constant curve nor with* γ_j *for any* $j \neq i$*.*

Remarks:

1. Although the effective majorizability is a very mild restriction on β, it cannot be removed. It is not difficult to see that for any M there exists an increasing function β_M such that any contractible closed geodesic γ on M can be contracted to a point by a homotopy passing only through closed curves of length $\leqq \beta_M(length(\gamma))$. (However, if the word problem for the fundamental group of M is unsolvable, then there is no effectively majorizable function β_M with this property. This fact was first observed by Gromov.)

2. Informally speaking, in our proof of Theorem 1 the geodesics γ_i arise as obstructions to fast solvability of the word problem for $\pi_1(M)$. The mentioned exponential lower bound for their number is due to the fact that not only the word problem for groups from the class Σ is algorithmically unsolvable, but this problem is unsolvable in recursive time using any amount of oracle information which grows subexponentially with the length of considered words.

3. Note that the constant $c(M)$ does not depend on β!

4. Assume that instead of the last statement of Theorem 1 one wants to prove only the following weaker statement: For some (arbitrary) *fixed* increasing effectively majorizable function β for all sufficiently large L there are $[c(M)^L]$ contractible closed geodesics $\gamma_1, \dots, \gamma_{[c(M)^L]}$ of length $\leqq L$ such that, for any i, γ_i can be connected with neither a constant geodesic nor with γ_j for some $j \neq i$ by a homotopy passing only through curves of length $\leqq \beta(length(\gamma_i))$. Then one can find also groups with solvable word problem for which this weaker statement is true.

5. It will be clear from the proof that the number of closed geodesics of length $\leqq l$ in any fixed homotopy class also grows at least exponentially with l.

6. Our proof of Theorem 1 can be used to write down specific examples of finite presentations of groups from Σ.

2. Time-bounded Kolmogorov complexity of the word problem for the fundamental group and contractible closed geodesics.

Recall that the resource-bounded Kolmogorov complexity of a finite binary sequence can be defined as follows (for more details see [14] , [22] , [7] , [4] , [17]): Consider an universal Turing machine T with an input tape, worktape(s) and an output tape. (We assume that the alphabet consists of three symbols : 0, 1 and the blank.) Let us say that a binary sequence x can be computed from a binary sequence y in time t and space s if T produces x on the output tape using not more than s squares on worktapes and in not more than t steps, when it starts its work with y on the input tape and empty worktapes and the output tape. The minimal length of a binary sequence y such that x can be computed from y in time t and space s will be denoted by $K_T^{t,s}(x)$. It is known that there exists an optimal universal Turing machine U with two worktapes such that for any universal Turing machine T and any binary sequence x

$$(*) \qquad K_U^{const\ t\ \ln t, const\ s}(x) \leqq K_T^{t,s}(x) + const,$$

where *const* denote constants depending only on U and T (cf. [17], p.240, [14]).

Now let $t(n)$, $s(n)$ be any increasing functions from $\mathbb{N}$ to $\mathbb{N}$. For any subset $P \subset \mathbb{N}$ we can regard its characteristic function κ_P as an infinite sequence of 0's and 1's. Denote this sequence by $S(P)$ and the sequence formed by the first n digits of $S(P)$ by $S^{(n)}(P)$. Now consider the function $K_U^{t(n),s(n)}(S^{(n)}(P))$. This function slightly depends on the choice of U. To avoid this problem we can introduce the following equivalence relations on the sets of functions from $\mathbb{N}$ to $\mathbb{N}$: for two time bounds t_1 and t_2, $t_1 =_{time} t_2$ if, for any n, $\max\{t_1(n)/t_2(n),\ t_2(n)/t_1(n)\} \leqq const(\max\{\ln t_1(n),\ \ln t_2(n)\})^{const}$; for two space bounds $s_1(n)$, $s_2(n)$ $s_1 =_{space} s_2$ if $\max\{s_1(n)/s_2(n),\ s_2(n)/s_1(n)\} \leqq const$, where *const* denotes constants not depending on n. We will regard the functions $t(n)$, $s(n)$ as representatives of the corresponding equivalence classes. Also consider the equivalence relation defined as follows: $f =_{Kolm} g$ if and only if there exists $c \in \mathbb{N}$ such that, for any n, $|f(n) - g(n)| \leqq c$. Consider the union of equivalence classes with respect to the relation $=_{Kolm}$ of all functions $K_U^{\bar{t}(n),\ \bar{s}(n)}(S^{(n)}(P))$, where $\bar{t}$ and $\bar{s}$ run over all functions from the equivalence classes $=_{time}$ of $t(n)$ and $=_{space}$ of $s(n)$, correspondingly. The formula (*) shows that the resulting set of functions does not depend on the particular choice of U. It will be denoted $K^{t,s}(P, n)$ and called *resource-bounded Kolmogorov complexity of the membership in P problem.* Sometimes we will use this term for specific functions from this class. If one does not impose any bounds on time (equivalently, we can formally let t be equal to ∞), or on space, or on neither time, nor space, then one obtains, correspondingly, *space-bounded Kolmogorov complexity*, *time-bounded Kolmogorov complexity* or just *Kolmogorov complexity* of the membership in P problem. In this paper we will be mostly interested in the time-bounded Kolmogorov complexity. From now on we will sometimes omit the superscript indicating the infinite space bound in the notation for the time-bounded Kolmogorov complexity.

The notion of time-bounded Kolmogorov complexity of the word problem of a finitely presented group G can be introduced as follows: First, fix a finite presentation F of G. Consider a numeration of the set of all words in generators of G and their inverses by consecutive integers such that numbers assigned to words increase with the length of words and such that words of the same length are numerated using a lexicographic order. The set of words representing the trivial element of G can be now regarded as a recursively enumerable subset N_F of $\mathbb{N}$.

Consider the time-bounded Kolmogorov complexity of the membership in N_F problem $K^t(N_F,\ n)$. Denote by *Gen* the number of generators in F. Now consider the class of functions $K_F^t \equiv K^{t\circ[\log_{2Gen}]}(N_F,\ (2Gen)^n + 1)$. (We substitute $(2Gen)^n + 1$ for the argument of this function because the first $(2Gen)^n + 1$

natural numbers correspond to the words of length $\leqq n$ of G, where Gen is the number of generators of G in the considered finite presentation. The purpose of taking the composition of t with $[\log_{2Gen}]$ is to reexpress the time bound which is supposed to be a function of the length of the word as a function of the number of the word in the considered numbering.) We will call this class of functions of n *time-bounded Kolmogorov complexity of the word problem for G in finite presentation F*. The union of all such classes K^t_F over the set of all finite presentations F of G will be called *time-bounded Kolmogorov complexity of the word problem for G* and denoted by $K^t(Word(G),\ n)$. (We will also use the same term for any specific function from this set of functions.) Observe, however, that if F_1 and F_2 are two finite presentations of the same group, then there exists a constant c such that for any function $\phi_1 \in K^t_{F_1}$ there exists a function $\phi_2 \in K^t_{F_2}$ such that $\phi_1(n) \leqq \phi_2(cn)$ for every n. So, informally speaking, all functions in the class $K^t(Word(G),\ n)$ have similar growth. Space-bounded or, more generally, resource-bounded Kolmogorov complexity of the word problem for a finitely presented group can be defined in a similar way. For any given functions t and s, $K^{t,s}(Word(G), n)$ indicates the growth of the minimal number of bits of oracle information necessary to solve the the word problem for G for any given word of length $\leqq n$ in time not exceeding $t(n)$ and using the amount of space on worktapes not exceeding $s(n)$, when $n \longrightarrow \infty$.

Note that for any universal machine U and any natural number t the time-bounded Kolmogorov complexity $K^{t,\ \infty}_U(x)$ of a binary sequence x is a Turing computable function of t and x. (Indeed, we can try one by one all binary sequences y. For every binary sequence y we can check whether or not x is computed from y in not more than t steps. However, note that the standard Kolmogorov complexity $K^{\infty,\infty}_U(x)$ is *not* Turing computable!) Hence Theorem 2.5 in [22] implies the following result of Barzdin ([2]): There exists a recursively enumerable set $E \subset \mathbb{N}$ and a positive number $Const$ such that for any Turing computable function $t(n)$ the time-bounded Kolmogorov complexity with time resources bounded by t of the memberhip in E problem is bounded from below by $\frac{n}{Const}$ (see also section 3 of [18]). (Theorem 2.5 in [22] states that there exists a recursively enumerable set E and a positive number $Const$ such that for any recursive majorant Φ of Kolmogorov complexity of binary sequences (without any bounds on resources) there exists a constant c such that, for any n, $\Phi(S^{(n)}(E)) \geqq n/Const - c$.) The proof of Theorem 2.5 in [22] is constructive. One can use this proof to write down an explicit (one-tape) Turing machine T_E such that T_E halts if and only if the input is an element of the set E. (A proof of a very similar result in [7] (Theorem 9) implies that for any specific Turing computable function $t(n)$ one can construct a *recursive* set $E_t \subset \mathbb{N}$ such that the time-bounded Kolmogorov complexity with time resources bounded by t of the membership in E_t problem is bounded from below by $n/const$. Similarly, one can use the idea of the proof of Theorem 9 in [7] to find a specific Turing machine T_t such that E_t is its halting set. This remark (together with the proof of Theorem 1 below) is the basis for Remark 4 after the text of Theorem 1.)

The classical proof of the algorithmic unsolvability of the triviality problem for finitely presented groups which can be found in [21] implies that it is possible to write down explicitly a finite presentation F_E of a group G_E with the following property: For any number $m \in \mathbb{N}$ one can indicate a word $w_m \in G_E$ of length not exceeding $const_0(\log_2 m + 1)$ such that $w_m = e$ in G_E if and only if $m \in E$. (Here $const_0$ is a specific constant which can be explicitly found. The appearance of the logarithm is due to the fact that when m is regarded as an input for the Turing machine T_E it is presented by a binary sequence of length $[\log_2\ m] + 1$.) Hence for any recursive function t and any function $\phi \in K^t_{F_E}(Word(G_E), n)$ $\phi(n) \geqq [\frac{2^{\frac{n}{const_0}}}{C}]$, where $C > 0$ is a constant depending on F_E, t and the choice of ϕ. Therefore, Theorem 1 directly follows from the following Proposition 2:

PROPOSITION 2. *For any compact Riemannian manifold M, any finite presentation F of $\pi_1(M)$ and any increasing effectively majorizable function β there exist a constant $c > 0$, a constant $C_F > 0$ depending only on F and M, and a Turing computable function $t(n)$ such that for a certain function $k(n)$ in the class $K^t_F = K^t_F(Word(\pi_1(M)),\ n)$ there exist at least $[\frac{k([\frac{l}{C_F}])}{c\ l}]$ geometrically distinct non-constant contractible closed geodesics γ_i of length not exceeding l on M such that, for any i, γ_i is not contractible to a constant loop via closed curves of length not exceeding $\beta(length(\gamma_i))$ and for any $i,\ j$ such that $i \neq j$ γ_i and γ_j cannot be connected by a homotopy passing through closed curves of length not exceeding $\beta(\max\{length(\gamma_i),\ length(\gamma_j)\})$.*

Important remark. If one is interested in the number of all geometrically distinct contractible non-constant closed geodesics γ of length $\leqq l$ on M and not only of those which cannot be contracted to a point by a homotopy passing only via closed curves of length $\leqq \beta(length(\gamma))$, then the lower bound given in the proposition can be easily improved. For example, it seems very plausible (although I did not check the details) that the number of contractible closed geodesics of length $\leqq l$ is greater than $k_0(l)/(const\ l)$, where $k_0(l)$ is some function from the equivalence class $K^{l(\ln l)^{const},\ l}(Word(\pi_1(M)),\ l)$. We postpone the explanation of this statement till the end of section 3 (see Remark after the proof of Proposition 2).

3. Proof of Proposition 2.

First, we are going to prove Proposition 2 assuming that M is a smooth semialgebraic submanifold of an Euclidean space defined as the set of solutions of a system of polynomial equations and inequalities such that all coefficients of polynomials in this system are algebraic. Thus, M can be represented by a finite set of data and this representation of M can be used in an algorithm we are going to describe below. Let β be any increasing effectively majorizable function. For any l define the equivalence relation $=_\beta$ on the set $Geod_0(l)$ of all non-constant geometrically distinct contractible closed geodesics of length $\leqq l$ M as follows: $\gamma_1 =_\beta \gamma_2$ if and only if there exists a homotopy between γ_1 and γ_2 passing through curves of length not exceeding $\beta(l)$. Denote the number of equivalence classes of $Geod_0(l)$ with respect to this relation by $m_{\beta,M}(l)$. Proposition 2 can be reformulated as the statement that for any finite presentation F of $\pi_1(M)$ and any increasing effectively majorizable function β there exists a constant c, a constant $C_F > 0$ not depending on β, and a Turing computable function t such that $clm_{\beta,M}(l)$ majorizes the time-bounded Kolmogorov complexity $K^t_F(Word(\pi_1(M)), [\frac{l}{C_F}])$ of the word problem for $\pi_1(M)$. To prove this statement it is sufficient to show the existence of an algorithm which solves the word problem for $\pi_1(M)$ for all words of length $\leqq n$ in a time bounded by a Turing computable function of n and using as an oracle information descriptions of one geodesic from every equivalence class of $Geod_0(const\ n)$ with respect to $=_\beta$ for some constant $const$. Here one is allowed to use only $Const\ l$ bits of information in order to represent a geodesic of length $\leqq l$. (In order to represent a geodesic γ_i using not more than $Const\ length(\gamma_i)$ bits of information we proceed as follows. Let $inj(M)$ denotes the injectivity radius of M. The geodesic γ_i will be (approximately) represented by a piecewise-geodesic curve $\bar\gamma_i$ passing through points $p_0, \dots, p_m$ from a certain fixed (i.e. not depending on i) $(inj(M)/1000)$-net NET on M. For any i the points p_j are defined as follows: We can assume that $\gamma_i : [0, length(\gamma_i)] \longrightarrow M$ is parametrized by the arclength. Let $m = [100 length(\gamma_i)/inj(M)]+1$. For every $j = 0, 1, \dots, m$ define p_j as the closest to $\gamma_i(j\frac{length(\gamma_i)}{m})$ point of NET. For any j, $\bar\gamma_i$ connects p_j and p_{j+1} via the (unique) shortest geodesic. Note that closed curves $\bar\gamma_i$ will be also contractible, that $length(\bar\gamma_i) \leqq 2 length(\gamma_i)$, and that, for any i, γ_i and $\bar\gamma_i$ can be connected by a homotopy passing through curves of length not exceeding $10l$. To be able to represent points from NET in a finite form we can assume that any point x of NET is the closest point of M to some point x^* with rational coordinates in a sufficiently small neighborhood of M in the ambient Euclidean space. The vector of coordinates of x^* will be used to represent $x \in NET$. It is clear that the amount of information required to represent any point of NET (and in particular p_j) will not exceed a constant depending on M and on a particular choice of NET but not on γ_i.) Here is a sketch of the algorithm (see section 4 of [19] for a more detailed description): We realize a given word by an embedded loop γ. Compute an integer upper bound L^* for the length of γ. For a sufficiently small positive ϵ construct a (finite) ϵ-net N_ϵ in the space of all closed curves of length $\leqq 100\beta(L^*)$ uniformly (with respect to the arclength) parametrized by $t \in [0,1]$. (The distance between two curves ρ_1 and ρ_2 is, by definition, $dist(\rho_1, \rho_2) = \max_{t\in[0,1]} dist_M(\rho_1(t),\ \rho_2(t))$.) The value of ϵ should be chosen rational and small enough to ensure that if for two closed curves $\rho_1,\ \rho_2$ $dist(\rho_1, \rho_2) \leqq 10\epsilon$, then ρ_1 and ρ_2 are homotopic. (It is easy to see that one can choose any rational $\epsilon < inj(M)/20$. For details see statement (d) in section 2 of [19]). Although a value of such ϵ can be computed, instead we can just assume that such a value is known to the algorithm. (This algorithm must work for a particular fixed M, so this value is just a constant.) The curves in the net N_ϵ can be chosen piecewise-geodesic and given by a finite sequence of points from a fixed sufficiently dense net on M. Then we form a graph Gr_ϵ such that its vertices correspond to curves from N_ϵ, and two vertices are connected by an edge if the corresponding curves are 3ϵ-close, and not connected if they are not 5ϵ-close. (If neither of these two possibilities occur, then the corresponding vertices can be either connected by an edge or not. Thus, we have a room for an error in approximate computations of distances.) Now find a closed curve $\rho \in N_\epsilon$ sufficiently close to γ as well as curves $\sigma_i \in N_\epsilon$ sufficiently close to $\bar{\gamma}_i$. Also consider a constant curve $\sigma_0 \in N_\epsilon$. It is clear that γ will represent the trivial element of $\pi_1(M)$ if and only if the vertex of Gr_ϵ corresponding to ρ is in the same component of Gr_ϵ as one of the vertices corresponding to σ_i, $i = 0, 1, \ldots, m_{\beta,M}(l)$.

To be more rigorous note that it is entirely obvious from the description of the algorithm how to realize it by a register machine, say, MRAM (see [8] , p.24, for the definition of MRAM. MRAM is a straightforward formalization of the notion of computation in an intuitive sense. For the reader not familiar with the definition of register machines note that any computer program written in a programming language such as FORTRAN, PASCAL or C and using only data of the integer type can be trivially rewritten as a MRAM). On the other hand any MRAM can be simulated by a Turing machine (with only a constant-factor space overhead) (cf. [8] , p.29). Thus, the above-described algorithm can be written as a Turing machine. This completes the proof in the case when M is isometric to a smooth semialgebraic submanifold of an Euclidean space defined as a set of solutions of a system of polynomial equations and inequalities such that all coefficients in this system are algebraic.

Now we are going to prove Proposition 2 in the general case. Note that by virtue of the Nash embedding theorem (cf. [12]) M can be isometrically embedded in an Euclidean space. By virtue of the Nash-Tognoli theorem (cf. [3], [16]) the isometric copy $\bar{M}$ of M can be approximated arbitrarily closely in C^2-norm by semialgebraic smooth submanifolds of an Euclidean space. The Tarski-Seidenberg theorem ([3]) implies that these manifolds can be chosen to be sets of solutions of systems of algebraic equations and inequalities where all coefficients are algebraic numbers (see [6] for details). (Actually, it seems that the fact that the approximating manifolds can be chosen as smooth semialgebraic or even algebraic sets defined over the field of real algebraic numbers or even over the field of rationals follows directly from the proof of the Nash-Tognoli theorem, although I did not check the details (see [1] , where a slightly weaker statement was proven).) The Cheeger inequality (cf. [5] , [9]) providing a lower bound for the injectivity radius of a Riemannian manifold in terms of its volume, diameter and the supremum of the absolute value of sectional curvatures implies that the injectivity radii of these approximating manifolds will be uniformly bounded from below by a positive constant. Thus, in particular, there exists a smooth semialgebraic submanifold A of an Euclidean space such that the Gromov-Hausdorff distance $d_{GH}(A,\ \bar{M})$ between A and $\bar{M}$ does not exceed $\min\{inj(A),\ inj(\bar{M})\}/100$, and such that A is a solution of a system of algebraic equations and inequalities where all coefficients are algebraic numbers. (Here we regard A and $\bar{M}$ as metric spaces with the inner metrics. The definition and simplest properties of the Gromov-Hausdorff metric on the space of all compact metric spaces can be found, for example, in [13] or [20] .) We have already proven that Proposition 2 holds for A. Hence Proposition 2 for $\bar{M}$ immediately follows from the inequality

$$m_{\beta^*,A}(l/2) \leqq m_{\beta,\bar{M}}(l),$$

where β^* is defined by the formula $\beta^*(l) = 40\beta(2l)$. Thus, it remains to prove this inequality.

We are going to use several simple observations which are proven in a slightly more general form in section 2 of [19] . First note that if $\rho_1, \rho_2 : [0,1] \longrightarrow T$ are two closed curves on a Riemannian manifold T such that for any t $dist_T(\rho_1(t),\ \rho_2(t)) \leqq inj(T)/4$, then ρ_1 and ρ_2 can be connected by a homotopy passing through curves of length not exceeding $10\max\{length(\rho_1),\ length(\rho_2)\}$. (This is a particular case of the statement (d) in section 2 of [19] ; see [19] for a detailed proof of this statement.) Further, let $M_1,\ M_2$ be two compact Riemannian manifolds such that the Gromov-Hausdorff distance between M_1 and M_2 does not exceed $\min\{inj(M_1),\ inj(M_2)\}/100$. Let $\tau : [0,1] \longrightarrow M_1$ be a closed parametrized curve. Choose an increasing sequence of points $t_1 = 0, t_1, \ldots, t_k < 1, t_{k+1} = 1$ in such a way that for every j the length of the segment of τ between $\tau(t_j)$ and $\tau(t_{j+1})$ is equal to $\frac{length(\tau)}{4[5length(\tau)/\min\{inj(M_1),\ inj(M_2)\}]+4}$. Now define $\bar{\tau} : [0,1] \longrightarrow M_2$ as follows: For any j, $\bar{\tau}(t_j)$ will be (one of) the closest point(s) of M_2 to $\tau(t_j)$. For every j connect $\bar{\tau}(t_j)$ and $\bar{\tau}(t_{j+1})$ by the shortest geodesic in M_2. Let us call the resulting curve a *transfer* of τ to M_2. If τ is not parametrized, then the transfer of its arbitrary parametrization will be called a transfer of τ. (Note, that in this case the transfer is not unique.) Similarly, we can define transfer of closed curves on M_2 to M_1. Observe, that if the length of τ is not less than $2inj(M_1)$, then the length of any transfer of τ to M_2 does not exceed $2length(\tau)$. (This fact can be easily deduced using just the triangle inequality. Note also that the length of any non-constant closed geodesic on M_1 is not less than $2inj(M_1)$.) Now observe that if $length(\tau) \geqq 2inj(M_1)$, then τ and a transfer to M_1 $\tilde{\tau}$ of a transfer to M_2 of τ can be connected by a homotopy passing through curves of length not exceeding $40\ length(\tau)$ since (a) $length(\tilde{\tau}) \leqq\ 4\ length(\tau)$; and (b) for any t $dist_{M_1}(\tau(t), \tilde{\tau}(t)) < inj(M_1)/4$. (Both (a) and (b) easily follow from the triangle inequality. If τ was not initially parametrized, then we must consider it with the parametrization chosen to make the transfer to M_2.) This observation can be used to demonstrate that if τ_1 and τ_2 are two closed geodesics (one of them can be constant) such that there exists a homotopy between their transfers to M_2 $\bar{\tau}_1$ and $\bar{\tau}_2$ which passes only through closed curves of length $\leqq L$, where $L \geqq 2\max\{inj(M_1),\ inj(M_2)\}$, then there exists a homotopy between τ_1 and τ_2 which passes only through curves of length $\leqq \max\{40length(\tau_1),\ 40length(\tau_2),\ 20L\}$ on M_1. This homotopy will consist of three pieces: a homotopy between τ_1 and the transfer $\tilde{\tau}_1$ of $\bar{\tau}_1$ to M_1 which passes through curves of length $\leqq\ 40length(\tau_1)$, a homotopy between $\tilde{\tau}_1$ and the transfer $\tilde{\tau}_2$ of $\bar{\tau}_2$ on M_1 which passes through curves of length $\leqq 20L$ and a homotopy between $\tilde{\tau}_2$ and τ_2 which passes through curves of length $\leqq 40length(\tau_2)$. The homotopies between τ_1 and $\tilde{\tau}_1$ and between τ_2 and $\tilde{\tau}_2$ were already explained above. To construct the homotopy between $\tilde{\tau}_1$ and $\tilde{\tau}_2$ consider the homotopy $H : [0,1] \longrightarrow \Omega\ M_2$ between $\bar{\tau}_1$ and $\bar{\tau}_2$ which passes through curves of length $\leqq L$. (Here $\Omega\ M_2$ denotes the space of parametrized piecewise-smooth closed curves on M_2.) Consider an increasing sequence $0 = s_1, s_2, \ldots, s_{k-1}, s_k = 1$ such that for any j and for any $t \in [0,1]$ $dist_{M_2}(H(s_j)(t),\ H(s_{j+1})(t)) < \min\{inj(M_1)\ inj(M_2)\}/100$. Now it is easy to see that for any j (a) the length of the transfer of $H(s_j)$ to M_1 does not exceed $2L$; and that (b) the transfers of $H(s_j)$ and of $H(s_{j+1})$ to M_1 can be connected by a homotopy passing through closed curves of length $\leqq 20L$. Patching all these homotopies together we obtain the required homotopy between $\tilde{\tau}_1$ and $\tilde{\tau}_2$.

Now we can prove the required inequality $m_{\beta^*,A}(l/2) \leqq\ m_{\beta,\bar{M}}(l)$, where $\beta^*(l) = 40\beta(2l)$. Without any loss of generality we can assume also that, for any l, $\beta(l) \geqq l/2$. Consider a representative from every equivalence class of closed geodesics of length $\leqq l/2$ on A with respect to the introduced above equivalence relation $=_{\beta^*}$. Denote these geodesics by γ_i, $i = 1, \ldots, m_{\beta^*,A}(l/2)$. It immediately follows from the discussion above that for any i, j such that $i \neq j$ the transfers $\bar{\gamma}_i$ and $\bar{\gamma}_j$ of γ_i and γ_j to $\bar{M}$ cannot be connected by a homotopy passing through closed curves of length $\leqq \frac{1}{40}\beta^*(l/2) = \beta(l)$. But for any i $\bar{\gamma}_i$ can be connected by a homotopy passing through curves of length $\leqq length(\bar{\gamma}_i)$ with a closed geodesic Γ_i on $\bar{M}$. The closed geodesics Γ_i for $i = 1, \ldots, m_{\beta^*,A}(l/2)$ obviously belong to different equivalence classes of the set of closed geodesics of length $\leqq l$ on $\bar{M}$ with respect to the equivalence relation $=_\beta$. This completes the proof of Proposition 2.

Remark. We beleive that one can strengthen Proposition 2 (at least in the important case when $\beta(l) = l/2$) using a version of the Birkhoff polygonal process or of one of known curve-shortening flows (cf. [15]) to improve the performance of the algorithm used in the proof described above of Proposition 2. In particular, it seems plausible that there exists a very fast algorithm using an amount of space on the worktapes which is linear in $length(\gamma)$ (and no oracle information) and checking whether or not a given closed curve γ on M can be connected with a constant loop or with one of the geodesics from the given set of all non-constant contractible closed geodesics on M of length $\leqq\ length(\gamma)$ by a homotopy passing only through closed curves of length not exceeding the length of γ. This idea is the motivation for the remark at the end of Section 2.

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Bibliography

[1] E. Ballico, A. Tognoli, "Algebraic models defined over Q of differentiable manifolds", Geom. Dedicata, 42(1992), 155-162.
[2] J.M. Barzdin, "Complexity of programs to determine whether natural numbers not greater than n belong to a recursively enumerable set", Soviet Math. Doklady 9(1968), 1251-1254.
[3] J. Bochnak, M. Coste, M.-F. Roy, "Géometrie Algébrique Réelle", Springer, 1987.
[4] G. Chaitin, "Information, Randomness and Incompleteness", Singapore, World Scientific, 1987.
[5] J. Cheeger, "Finiteness theorems for Riemannian manifolds", Amer. J. Math. 92(1970), 61-74.
[6] M. Coste, M. Shiota, "Nash triviality in families of Nash manifolds", Invent. Math. 108(1992), 349-368.
[7] R.P. Daley, "Minimal program complexity of sequences with restricted resources",Inf. and Control 23(1973), 301-312.
[8] van Emde Boas, "Machine Models and Simulations", in "Handbook of Theoretical Computer Science", ed. by J. van Leeuwen, Elsevier, 1990, pp. 1-66.
[9] K. Fukaya, "Hausdorff convergence of Riemannian manifolds and its applications", in "Recent topics in Differential and Analytic Geometry", ed. T. Ochiai, Kinokuniya, 1990.
[10] M. Gromov, "Hyperbolic manifolds, groups and actions", in "Riemannian surfaces and related topics", (I. Kra and B. Maskit, ed.), Ann. of Math. Studies 97(1981), pp. 183-215.
[11] M. Gromov, "Asymptotic invariants of infinite groups", in "Geometric group theory", vol. 2, eds. G. Niblo and M. Roller, Cambridge University Press, 1993.
[12] M.L. Gromov, V.A. Rokhlin, "Embeddings and immersions in Riemannian geometry", Russian Math. Surv., 25(5)(1970), 1-57.
[13] K. Grove, "Metric differential geometry", in "Differential Geometry", ed. V. Hansen, Springer Lecture Notes in Mathematics 1263, 171-227, Springer, 1987.
[14] J. Hartmanis, "Generalized Kolmogorov Complexity and the structure of feasible computations", in "24th Annual Symposium on Foundations of Computer Science:, IEEE Computer Society Press, 1984, pp.439-445.
[15] J. Hass, P. Scott, "Shortening curves on surfaces", Topology, 33(1994), 25-43.
[16] N.V. Ivanov, "Approximation of smooth manifolds by real algebraic sets", Russ. Math. Surveys 37(1982), 1-59.
[17] M. Li, P.M.B. Vitanyi, "Kolmogorov Complexity and its applications", in "Handbook of Theoretical Computer Science", ed. by J. van Leeuwen, Elsevier, 1990, pp. 187-254.
[18] A. Nabutovsky, "Geometry of the space of triangulations of a compact manifold", to appear in Comm. Math. Phys.
[19] A. Nabutovsky, "Disconnectedness of sublevel sets of some Riemannian functionals", to appear in Geom. Funct. Anal.
[20] P. Petersen V, "Gromov-Hausdorff convergence of metric spaces", in "Differential Geometry: Riemannian Geometry", ed. R. Greene and S.T. Yau, Proceedings of AMS Symposia in Pure Math., vol. 54, part 3, 489-504, 1993.
[21] J.J. Rotman, "An introduction to the theory of groups", Allyn and Bacon, Boston, 1984.
[22] A.K. Zvonkin, L.A. Levin, "The complexity of finite objects and the development of concepts of information and randomness by means of the theory of algorithms", Russian Math. Survey 25(6)(1970), 83-129.

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