Combining Determinism and Non-Determinism

Michael Stephen Fiske

Aemea Institute
San Francisco, California, 94129

Abstract. Our goal is to find mathematical operations that combine non-determinism measured from quantum randomness with computational determinism so that non-mechanistic behavior is preserved in the computation. Formally, some lemmas about operations applied to computably enumerable (c.e.) sets and bi-immune sets are proven here.

1 Introduction

In [2], a lemma about the symmetric difference operator applied to a computably enumerable set and bi-immune set is stated without proof. A proof of this lemma is provided.

1.1 Notation and Conventions

$\mathbb{N}$ is the non-negative integers. $\mathbb{N}_+$ is $\mathbb{N}-\{0\}$. $\mathbb{E}$ is the even, non-negative integers.

Let $x_0\ x_1\ \ldots \in \{0,1\}^{\mathbb{N}}$ be a binary sequence. The sequence $x_0\ x_1\ \ldots$ induces a set $\mathcal{A} \subset \mathbb{N}$, where the identification of $\mathcal{A}$ with the sequence means $k \in \mathcal{A}$ if and only if $x_k = 1$.

$\overline{\mathcal{A}}$ is the complement of set $\mathcal{A}$. The relative complement is $\mathcal{A} - \mathcal{B} = \{x \in \mathcal{A} : x \notin \mathcal{B}\}$. $\oplus$ is exclusive-or: $0 \oplus 0 = 1 \oplus 1 = 0$ and $1 \oplus 0 = 0 \oplus 1 = 1$. Because of the identification between a binary sequence and a subset of $\mathbb{N}$, the symmetric difference of $\mathcal{A}$ and $\mathcal{B}$ is represented as $\mathcal{A} \oplus \mathcal{B} = (\mathcal{A} - \mathcal{B}) \cup (\mathcal{B} - \mathcal{A})$, instead of the usual Δ. Herein $\oplus$ never represents the *join* operation as used in [1].

2 Preserving Non-Mechanistic Behavior

Our goal is to find operations that combine non-determinism measured from quantum randomness [3] with computational determinism [5] so that the non-mechanistic behavior (bi-immunity) is preserved. The following two lemmas formally address our intuition.

From [4], recall the definition of an immune and bi-immune set.

Definition 1. *Let $\mathcal{A} \subset \mathbb{N}$. Set $\mathcal{A}$ is* **immune** *if conditions (i) and (ii) hold.*
(i) $\mathcal{A}$ is infinite.
(ii) For all $\mathcal{B} \subset \mathbb{N}$, ($\mathcal{B}$ is infinite and computably enumerable) $\implies$ $\mathcal{B} \cap \overline{\mathcal{A}} \neq \emptyset$

Set $\mathcal{A}$ is **bi-immune** *if both $\mathcal{A}$ and $\overline{\mathcal{A}}$ are immune.*

Lemma 1. *Suppose $\mathcal{A}$ is bi-immune. Let $\mathcal{R}$ be a finite set. Then $\mathcal{A} \cup \mathcal{R}$ and $\mathcal{A} - \mathcal{R}$ are both bi-immune.*

PROOF. Set $\mathcal{A}^+ = \mathcal{A} \cup \mathcal{R}$ and $\mathcal{A}^- = \mathcal{A} - \mathcal{R}$. From definition 1, $\mathcal{A}^+$ and $\mathcal{A}^-$ are still infinite because a finite number of elements have been added to or removed from $\mathcal{A}$, respectively. For condition (ii), suppose there exists a computably enumerable set $\mathcal{B}$ such that $\mathcal{B} \cap \overline{\mathcal{A}^+} = \emptyset$. Then $\mathcal{B} \cap \overline{\mathcal{A}^+}$ and $\mathcal{B} \cap \overline{\mathcal{A}}$ only differ on a finite number of elements which implies that a c.e. set $\mathcal{B}'$ can be constructed from $\mathcal{B}$ such that $\mathcal{B}' \cap \overline{\mathcal{A}} = \emptyset$. Contradiction. The same argument holds for $\mathcal{A}^-$. □

Lemma 1 doesn't hold if $\mathcal{R}$ is an infinite, computable set: $\mathcal{A} \cup \mathbb{E}$ is not even immune as $\mathbb{E} \cap (\overline{\mathcal{A} \cup \mathbb{E}}) = \emptyset$. While the isolated operations of union and relative complement don't preserve bi-immunity, the symmetric difference operation does preserve it.

Lemma 2. *If $\mathcal{R}$ is c.e. and $\mathcal{A}$ is bi-immune, then $\mathcal{A} \oplus \mathcal{R}$ is bi-immune.*

PROOF. *Condition (i).* Verify that $\mathcal{A} \oplus \mathcal{R}$ is infinite. For the case that $\mathcal{R}$ is finite, let K be the largest element in $\mathcal{R}$. Then $\mathcal{A} \oplus \mathcal{R}$ is the disjoint union of the finite set $\{x \in \mathcal{A} \oplus \mathcal{R} : x \leq K\}$ and the infinite set $\{x \in \mathcal{A} : x > K\}$ since $\mathcal{A}$ is bi-immune.

Otherwise, $\mathcal{R}$ is infinite. By contradiction, suppose $\mathcal{A} \oplus \mathcal{R}$ is finite. Because $\mathcal{A}$ is the disjoint union of $\mathcal{A} - \mathcal{R}$ and $\mathcal{A} \cap \mathcal{R}$, then $\mathcal{A}$'s bi-immunity implies that $\mathcal{A} \cap \mathcal{R}$ is infinite. Also, let $\mathcal{R} - \mathcal{A} = \{r_1, r_2 \ldots, r_m\}$ since it is finite.

Claim: $\mathcal{A} \cap \mathcal{R}$ is c.e. Consider Turing machine M that enumerates $\mathcal{R}$. Concatenate the following machine N to machine M. Machine N does not halt if M halts with r_k where $1 \leq k \leq m$. For all other outputs where M halts, after machine N checks that M's output is not in $\mathcal{R} - \mathcal{A}$, then N immediately halts.

Now $\mathcal{A} \cap \mathcal{R}$ is infinite and c.e., contradicting $\mathcal{A}$'s bi-immunity, so $\mathcal{A} \oplus \mathcal{R}$ is infinite.

Condition (ii). Set $\mathcal{Q} = \mathcal{A} \oplus \mathcal{R}$.

By contradiction, suppose there exists an infinite, c.e. set $\mathcal{B}$ with $\mathcal{B} \cap \overline{\mathcal{Q}} = \emptyset$. Then $\mathcal{B} \subset \mathcal{Q}$. Also, $\mathcal{B} \cap \mathcal{R}$ is c.e. Now $\mathcal{B} \cap \mathcal{R} \subset \mathcal{R} - \mathcal{A} \subset \overline{\mathcal{A}}$ which contradicts that $\mathcal{A}$ is bi-immune, if $\mathcal{B} \cap \mathcal{R}$ is infinite. Otherwise, $\mathcal{B} \cap \mathcal{R}$ is finite. Set $K = max(\mathcal{B} \cap \mathcal{R})$. Then $\mathcal{B} \cap (\mathcal{A} - \mathcal{R})$ is infinite. Define the infinite, c.e. set $\mathcal{B}' = \{x \in \mathcal{B} : x > K\}$. Thus, $\mathcal{B}' \subset \mathcal{A}$ contradicts $\mathcal{A}$'s bi-immunity.

Similarly, by contradiction, suppose there exists infinite, c.e. set $\mathcal{B}$ with $\mathcal{B} \cap \mathcal{Q} = \emptyset$. Then $\mathcal{B} \subset \overline{\mathcal{Q}}$. Also, $\mathcal{B} \cap \mathcal{R}$ is c.e. Thus, $\mathcal{B} \cap \mathcal{R} \subset \mathcal{A} \cap \mathcal{R}$ which contradicts that $\mathcal{A}$ is bi-immune if $\mathcal{B} \cap \mathcal{R}$ is infinite. Otherwise, $\mathcal{B} \cap \mathcal{R}$ is finite. Set $K = max(\mathcal{B} \cap \mathcal{R})$. Define the infinite, c.e. set $\mathcal{B}' = \{x \in \mathcal{B} : x > K\}$. Thus, $\mathcal{B}' \subset \overline{\mathcal{A}}$, which contradicts $\mathcal{A}$'s bi-immunity. □

References

1. Rodney Downey and Denis Hirschfeldt. Algorithmic Randomness and Complexity. Springer-Verlag (2010).
2. Michael Stephen Fiske. Quantum Random Active Element Machine. Unconventional Computation and Natural Computation. LNCS 7956. Springer, 2013, pp. 252-254.
3. Miguel Herrero-Collantes and Juan Carlos Garcia-Escartin. Quantum random number generators. Reviews of Modern Physics. **89**(1), 015004, APS, Feb. 22, 2017.
4. Hartley Rogers, Jr. Theory of Recursive Functions and Effective Computability. MIT Press. (1987).
5. Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. Proc. London Math. Soc. Series 2 **42** (Parts 3 and 4), 230–265 (1936). A correction, ibid. **43**, 544–546 (1937).