if the graph of the function $(f \circ g \circ g \circ f)^m$ is transverse to the diagonal, then a point of period $4m$ is stable.

THEOREM 2.4. *Suppose $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are smooth. Generically, the set $\{x \in \mathbb{R}^n : g \circ f(x) = f \circ g(x) = x\}$ is empty. Precisely, for any open neighborhood U (C^∞ Whitney topology) about (f, g) in $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we can find smooth functions $(f_1, g_1) \in U$ such that the set $\{x \in \mathbb{R}^n : g_1 \circ f_1(x) = f_1 \circ g_1(x) = x\}$ is empty, and these smooth functions (f_1, g_1) are a residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.*

Proof: Let x represent n coordinates in $\mathbb{R}^n$. Let u represent n coordinates in $\mathbb{R}^n$. Let v represent n coordinates in $\mathbb{R}^n$. For each i, let z_i represent n coordinates in $\mathbb{R}^n$. Define the submanifold Σ of $\mathbb{R}^{9n}$ as $\Sigma = \{(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9) \in \mathbb{R}^{9n} : z_2 = z_4, z_3 = z_7, z_6 = z_8$, and $z_1 = z_8\}$. Notice each equation $z_2 = z_4, z_3 = z_7, z_6 = z_8$, and $z_1 = z_8$ represents n independent equations. Since there are no dependencies among these equations, there are $4n$ independent equations. Thus, the codimension of Σ in $\mathbb{R}^{9n}$ equals $4n$. Consider any smooth function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{2n}$, where $\Psi(x) = (f(x), g(x))$. Consider the multijet map $j_3{}^0\Psi(x, u, v) = (x, \Psi(x), u, \Psi(u), v, \Psi(v)) = (x, f(x), g(x), u, f(u), g(u), v, f(v), g(v))$. Notice that $(j_3{}^0\Psi)^{-1}(\Sigma) = \{(x, u, v) \in \mathbb{R}^{3n} : u = f(x), v = g(x)$, and $f \circ g(x) = g \circ f(x) = x\}$. If $(j_3{}^0\Psi)$ is transverse to the manifold Σ, then the previous set is diffeomorphic to the set $\{x \in \mathbb{R}^n : g \circ f(x) = f \circ g(x) = x\}$. Since the codimension of Σ is $4n$, and the dimension of the domain is $3n$, if $(j_3{}^0\Psi)$ is transverse to the manifold Σ, then Proposition 2.2 implies that $(j_3{}^0\Psi)^{-1}(\Sigma)$ is an empty set. The Multijet Transversality Theorem implies that $T_\Sigma = \{\Psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^{2n}) : j_3{}^0\Psi \pitchfork \Sigma\}$ is a residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{2n})$. From proposition 2.4, there is a homeomorphism H (w.r.t. to the C^∞ topology) from $C^\infty(\mathbb{R}^n, \mathbb{R}^{2n})$ to $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Since $C^\infty(\mathbb{R}^n, \mathbb{R}^{2n})$ is a Baire space, T_Σ is a dense subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{2n})$. Hence, $H(T_\Sigma)$ is a dense, residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. ■

Before we discuss the meaning of the following corollary, it is important to recall the difference between the periodicity of the non-autonomous system $\{f, g, f, g, f, g, \ldots\}$, in this case period 2, and the periodicity of an orbit, $[g \circ f \circ g \circ f]^m(p) = p$ for all $m \in \mathbb{N}$, which in this case is period 4. This Corollary says that for a non-autonomous system with period 2 that all points with period three are unstable. Notice that once we define the proper submanifold Σ, the proof uses the same argument as Theorem 2.4.

THEOREM 2.5. *Suppose $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are smooth. Generically, the set $\{x \in \mathbb{R}^n : f \circ (g \circ f)(x) = g \circ (f \circ g)(x) = x\}$ is empty. Precisely, for any open neighborhood U about (f, g) in the C^∞ Whitney topology, $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we can find smooth functions $(f_1, g_1) \in U$ such that the set $\{x \in \mathbb{R}^n : f_1 \circ (g_1 \circ f_1)(x) = g_1 \circ (f_1 \circ g_1)(x) = x\}$ is empty, and these smooth functions (f_1, g_1) are a residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.*

Proof: For each i, where $1 \leq i \leq 5$, let x_i represent n coordinates in $\mathbb{R}^n$. For each i, where $1 \leq i \leq 15$, let z_i represent n coordinates in $\mathbb{R}^n$. Define the submanifold Σ of $\mathbb{R}^{15n}$ as $\Sigma = \{(z_1, z_2, z_3, \ldots, z_{13}, z_{14}, z_{15}) \in \mathbb{R}^{15n} : z_2 = z_4, z_3 = z_7, z_8 = z_{10}, z_6 = z_{13}, z_1 = z_{12}$, and $z_1 = z_{14}\}$. Notice each equation represents n independent equations. Since there are no dependencies among these equations, there are $6n$ independent equations. Thus, the codimension of Σ in $\mathbb{R}^{9n}$ equals $6n$. Consider any smooth function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{2n}$, where $\Psi(x) = (f(x), g(x))$. Consider the multijet map $j_5{}^0\Psi(x_1, x_2, x_3, x_4, x_5) = (x_1, \Psi(x_1), x_2, \Psi(x_2), x_3, \Psi(x_3), x_4, \Psi(x_4), x_5, \Psi(x_5))$ $= (x_1, f(x_1), g(x_1), x_2, f(x_2), g(x_2), x_3, f(x_3), g(x_3), x_4, f(x_4), g(x_4), x_5, f(x_5), g(x_5))$. Notice that $(j_5{}^0\Psi)^{-1}(\Sigma) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^{5n} : x_2 = f(x_1), x_3 = g(x_1), x_4 = f \circ g(x_1), x_5 = g \circ f(x_1)$, and $g \circ f \circ g(x_1) = f \circ g \circ f(x_1) = x_1\}$. If $(j_5{}^0\Psi)$ is transverse to the manifold Σ, then the previous set is diffeomorphic to the set $\{x_1 \in \mathbb{R}^n : g \circ f \circ g(x_1) = f \circ g \circ f(x_1) = x_1\}$. Since the codimension of Σ is $6n$, and the dimension of the domain is $5n$, if $(j_5{}^0\Psi)$ is transverse to the manifold Σ, then Proposition 2.2 implies that $(j_5{}^0\Psi)^{-1}(\Sigma)$ is an empty set. The Multijet Transversality Theorem implies that $T_\Sigma = \{\Psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^{2n}) : j_5{}^0\Psi \pitchfork \Sigma\}$ is a residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{2n})$. From proposition 2.4, there is a homeomorphism H (w.r.t. to the C^∞ topology) from $C^\infty(\mathbb{R}^n, \mathbb{R}^{2n})$ to $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Since $C^\infty(\mathbb{R}^n, \mathbb{R}^{2n})$ is a Baire space, T_Σ is a dense, subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{2n})$. Hence, $H(T_\Sigma)$ is a dense, residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. ■

THEOREM 2.6. *Suppose $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are smooth. Generically, the set $\{x \in \mathbb{R}^n : f \circ (g \circ f)^k(x) = g \circ (f \circ g)^k(x) = x\}$ is empty. Precisely, for any open neighborhood U about (f, g) in the C^∞ Whitney topology, $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we can find smooth functions $(f_1, g_1) \in U$ such that the set $\{x : f_1 \circ (g_1 \circ f_1)^k(x) = g_1 \circ (f_1 \circ g_1)^k(x) = x\}$ is empty, and these smooth functions (f_1, g_1) are a residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.*

Proof: Similar argument as in Theorem 2.5.

THEOREM 2.7. *Suppose* $g_1, g_2, g_3 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *are smooth. Generically, the set* $\{x \in \mathbb{R}^n : g_2 \circ g_1(x) = g_1 \circ g_3(x) = g_3 \circ g_2(x) = x\}$ *is empty. Precisely, for any open neighborhood* U *(*C^∞ *Whitney topology) about* (g_1, g_2, g_3) *in* $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$*, we can find smooth functions* $(f_1, f_2, f_3) \in U$ *such that the set* $\{x \in \mathbb{R}^n : f_2 \circ f_1(x) = f_1 \circ f_3(x) = f_3 \circ f_2(x) = x\}$ *is empty, and these smooth functions* (f_1, f_2, f_3) *are a residual subset of* $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

Proof: For each i, where $1 \leq i \leq 4$, let x_i represent n coordinates in $\mathbb{R}^n$. For each i, where $1 \leq i \leq 16$, let z_i represent n coordinates in $\mathbb{R}^n$. Define the submanifold Σ of $\mathbb{R}^{16n}$ as $\Sigma = \{(z_1, z_2, z_3, \ldots, z_{14}, z_{15}, z_{16}) \in \mathbb{R}^{16n} : z_2 = z_5, z_3 = z_9, z_4 = z_{13}, z_7 = z_{12}, z_{12} = z_{14}$, and $z_1 = z_{14}\}$. Notice each equation represents n independent equations. Since there are no dependencies among these equations, there are $6n$ independent equations. Thus, the codimension of Σ in $\mathbb{R}^{16n}$ equals $6n$. Consider any smooth function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{3n}$, where $\Psi(x) = (g_1(x), g_2(x), g_3(x))$. Consider the multijet map $j_4{}^0\Psi(x_1, x_2, x_3, x_4) = (x_1, \Psi(x_1), x_2, \Psi(x_2), x_3, \Psi(x_3), x_4, \Psi(x_4)) = (x_1, g_1(x_1), g_2(x_1), g_3(x_1), x_2, g_1(x_2), g_2(x_2), g_3(x_2), \ldots, x_4, g_1(x_4), g_2(x_4), g_3(x_4))$. Notice that $(j_4{}^0\Psi)^{-1}(\Sigma) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^{4n} : x_2 = g_1(x_1), x_3 = g_2(x_1), x_4 = g_3(x_1)$, and $g_2 \circ g_1(x_1) = g_1 \circ g_3(x_1) = g_3 \circ g_2(x_1) = x_1\}$. If $(j_4{}^0\Psi)$ is transverse to the manifold Σ, then the previous set is diffeomorphic to the set $\{x \in \mathbb{R}^n : g_2 \circ g_1(x) = g_1 \circ g_3(x) = g_3 \circ g_2(x) = x\}$. Since the codimension of Σ is $6n$, and the dimension of the domain is $4n$, if $(j_4{}^0\Psi)$ is transverse to the manifold Σ, then Proposition 2.2 implies that $(j_4{}^0\Psi)^{-1}(\Sigma)$ is an empty set. The Multijet Transversality Theorem implies that $T_\Sigma = \{\Psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^{3n}) : j_4{}^0\Psi \pitchfork \Sigma\}$ is a residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{3n})$. From proposition 2.4, there is a homeomorphism H (w.r.t. to the C^∞ topology) from $C^\infty(\mathbb{R}^n, \mathbb{R}^{3n})$ to $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Since

$C^\infty(\mathbb{R}^n, \mathbb{R}^{3n})$ is a Baire space, T_Σ is a dense subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{3n})$. Hence, $H(T_\Sigma)$ is a dense, residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. ■

THEOREM 2.8. *Suppose m is an integer greater than 1. Suppose $g_1, \dots, g_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are smooth. Suppose $(\mathbb{R}^n, \{g_1, g_2, \dots, g_m, \dots\})$ is a non-autonomous system with period m. Suppose k is not a multiple of m. Then, generically the set $\{x \in \mathbb{R}^n : x = g_k \dots g_2 \circ g_1(x) = g_{2k} \cdots \circ g_{k+1}(x) = \cdots = g_{qk} \circ g_{qk-1} \circ \dots g_{(q-1)k+1}(x)$ for all positive integers $q\}$ is empty, and these smooth functions $(g_1, g_2, \dots, g_m)$ are a residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)^m$.*

Proof: Same argument as in Theorem 2.7.

We finish this section with a few remarks which summarize this section. We first require a definition.

DEFINITION 2.15. *Suppose $(X, \{f_1, f_2, \dots\})$ is a non-autonomous dynamical system. Then $(X, \{f_1, f_2, \dots\})$ is C^r structurally stable if there exists a continuous function $\epsilon : X \longrightarrow \mathbb{R}^+$ satisfying the following: if for any non-autonomous system $(X, \{g_1, g_2, \dots\})$, for each i, g_i lies in $B_\epsilon(f_i) = \{g \in C^\infty(X, X)$: for all $x \in X, d(j^r f_i(x), j^r g(x)) < \epsilon(x)\}$, then $(Y, \{g_1, g_2, \dots\})$ is dynamically equivalent to $(X, \{f_1, f_2, \dots\})$.*

REMARK 2.5. *Let M be a compact smooth n dimensional manifold. Suppose the non-autonomous system with period m $(M, \{f_1, f_2, \dots, f_m\})$ has a periodic point p*

with period k. Suppose k is not a multiple of m. Then $(M, \{f_1, f_2, \ldots, f_m\})$ is not structurally stable.

REMARK 2.6. *Suppose we have a non-autonomous system $(M, \{g_1, g_2, \ldots, g_m\})$ with period m. A way to find periodic points that are stable with respect to the non-autonomous system is to use a technique analogous to the poincare return map. Set $F = g_m \circ g_{m-1} \circ \cdots \circ g_2 \circ g_1$. Hence, $F \circ F = g_m \circ g_{m-1} \circ \cdots \circ g_2 \circ g_1 \circ g_m \circ g_{m-1} \circ \cdots \circ g_2 \circ g_1$, and so on. Thus, every m iterates of the non-autonomous system $(M, \{g_1, g_2, \ldots, g_m\})$ correspond to one iterate of the autonomous system (M, F). Hence, if the graph of F^k is transverse to the diagonal, then points with period k are stable.*

REMARK 2.7. *For any non-autonomous system $\{f_1, f_2, \ldots, f_m, f_1, f_2, \ldots, f_m, f_1, f_2, \ldots, f_m, \ldots\}$ period $m > 1$, all fixed points are unstable.*

Consequently, if the goal is to find training algorithms that converge toward a fixed point i.e. the fixed point represents a point where the network performs optimally, then we must impose additional hypotheses on the training functions.