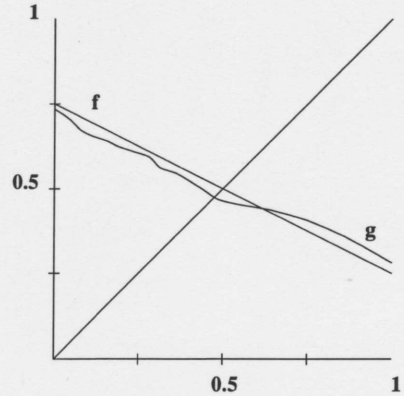
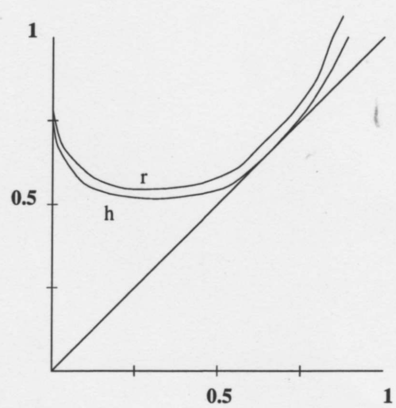
Instability of periodic points in non-autonomous systems

When modelling a physical system, we want to rely only on concepts or properties which remain unchanged if we use slightly different functions from the "actual modelling" functions. For example, one might prove that a particular training algorithm under certain conditions will always have a periodic orbit. However, if this periodic orbit could be annihilated, by arbitrarily small changes in the functions in the non-autonomous system, then in actual implementations of this training algorithm, we can not expect the periodic orbit to exist, or utilize the periodic orbit, when modelling physical systems with these functions.

To illustrate this point with an example, define the function $f : [0,1] \longrightarrow [0,1]$ as $f(x) = \frac{3}{4} - \frac{1}{2}x$. Notice that f has a fixed point at $\frac{1}{2}$. If we deform f just slightly to a new function g, then g may not have the fixed point at exactly $\frac{1}{2}$, but it will still have a fixed point.



In this case, the fixed point $x = \frac{1}{2}$ of f is stable under a small deformation of f. On the other hand, the fixed point of the function h shown in the following diagram is not stable.



By slightly deforming h to r, then a fixed point no longer exists, so the fixed point is unstable under a small deformation of h. We now introduce many definitions, and theorems that are developed in [GOLUB].

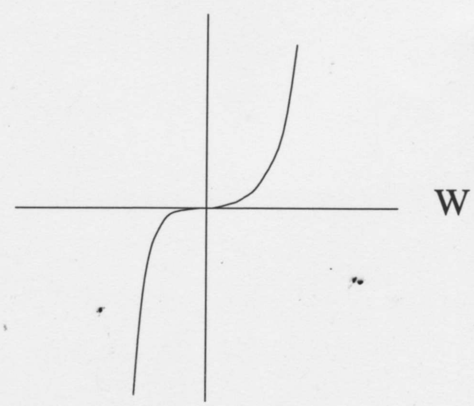
DEFINITION 2.9. *Let X, Y be smooth manifolds and $f : X \longrightarrow Y$ be a smooth mapping. Let W be a submanifold of Y and x a point in X. Then f intersects W*

transversely at (denoted by $f \,\overline{\pitchfork}\, W$ *at* x*) if either*

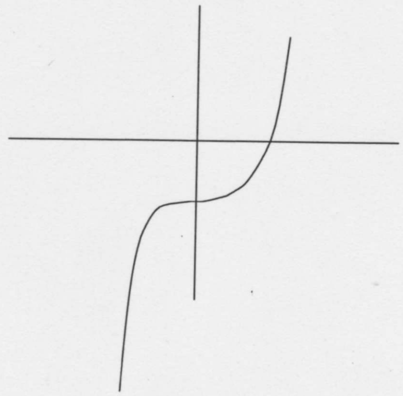
(I.) $f(x) \notin W$ *or*

(II.) $f(x) \in W$ *and* $T_{f(x)}Y = T_{f(x)}W + (df)_x(T_xX)$. *If* A *is a subset of* X*, then* f *intersects* W *transversely on* A *(denoted by* $f \,\overline{\pitchfork}\, W$ *on* A *if* $f \,\overline{\pitchfork}\, W$ *at* x *for all* $x \in A$*.) Finally,* f *intersects* W *transversely (denoted by* $f \,\overline{\pitchfork}\, W$ *) if* $f \,\overline{\pitchfork}\, W$ *on* X*.*

Let $X = \mathbb{R}$, $\mathbb{R} \times \{0\} = W$, $Y = \mathbb{R}^2$ and $f(x) = (x, x^3)$. Then $f \,\overline{\pitchfork}\, W$ at all nonzero x, but the intersection at $x = 0$ is a non-transverse intersection.



If we set $g(x) = (x, x^3 - 1)$, then $g \,\overline{\pitchfork}\, W$ at all x.



If we refer back to our original examples, we observe that f did not lose its fixed point when we perturbed it to g because the graph of f as a submanifold of $\mathbb{R}^2$ intersects the line $y = x$ transversely. On the other hand, when we deform the function h to the function r, the fixed point of h vanished because the graph of h does not transversely intersect the diagonal $y = x$ in $\mathbb{R}^2$.

DEFINITION 2.10. *If W is a submanifold of Y, the codimension of W equals the dimension of Y minus the dimension of W i.e. codim W = dim Y − dim W.*

The next proposition and theorem play a crucial role in showing that certain types of periodic points are unstable in non-autonomous systems.

PROPOSITION 2.2. *Let X and Y be smooth manifolds, and $W \subset Y$ a submanifold. Suppose dim W + dim X < dim Y i.e. the dim X < codim W. Let $f : X \longrightarrow Y$ be smooth and suppose that $f \pitchfork W$. Then $f(X) \cap W = \emptyset$.*

Proof: [GOLUB].

THEOREM 2.2. *Let X and Y be smooth manifolds, and W a submanifold of Y. Let $f : X \longrightarrow Y$ be smooth and assume that $f \pitchfork W$. Then $f^{-1}(W)$ is a submanifold of X. Also, $codim f^{-1}(W) = codim\, W$.*

Proof: [GOLUB].

We next introduce the definition of a jet bundle as presented in [GOLUB]. We develop these definitions because given a smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ we want to study the jet map $j^0 f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$, where for any $p \in \mathbb{R}^n$ we have $j^0 f(p) = (p, f(p))$. Notice that $j^0 f$ is just the graph of f. For completeness the definition below also develops higher order jets, but we do not use utilize them in later proofs.

DEFINITION 2.11. *Let X and Y be smooth manifolds and let p be a point in X. Suppose $f, g : X \longrightarrow Y$ are smooth maps with $f(p) = g(p) = q$.*

(I). f has first order contact with g at p if $(df)_p = (dg)_p$ as mappings of $T_pX \to T_qY$.

(II). f has kth order contact with g at p if $(df) : TX \longrightarrow TY$ has $(k-1)$st order contact with (dg) at every point in T_pX. This is written as $f \underset{k}{\sim} g$ at p. (k is a positive integer.)

(III). Let $J^k(X,Y)_{p,q}$ denote the set of equivalence classes under "$\underset{k}{\sim}$ at p" of mappings $f : X \longrightarrow Y$ where $f(p) = q$.

(IV). Let $J^k(X,Y) = \bigcup_{(p,q)\in X\times Y} J^k(X,Y)_{p,q}$. (Disjoint union). An element σ in $J^k(X,Y)$ is called a k-jet of mappings from X to Y.

(V). Let σ be a k-jet, then there exist p in X and q in Y for which σ is in $J^k(X,Y)_{p,q}$.

The point p is called the source of σ. *The mapping* $\alpha : J^k(X,Y) \longrightarrow X$ *given by* $\sigma \rightarrow$ (*source of* σ) *is the source map.*

Given a smooth mapping $f : X \longrightarrow Y$ there is a canonically defined mapping $j^k f : X \longrightarrow J^k(X,Y)$ called the k-jet of f defined by $j^k f(p)$ equals the equivalence class of f in $J^k(X,Y)_{p,f(p)}$ for every $p \in X$. In [GOLUB], they show that $j^k f(p)$ is a way, independent of the parameterization of the manifold, of representing the Taylor expansion of f at p up to order k. In the simpler case where $X = \mathbb{R}$, and $Y = \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$, then $j^k f$ describes the Taylor expansion as follows. We have $j^k f(p) = (p, f(p), f'(p), f^{(2)}(p), \ldots, f^{(k)}(p))$. For our purposes we only use $j^0 f$ which is the graph of f.

In general, $J^k(X,Y)$ is a smooth manifold. We only consider $J^0(\mathbb{R}^n, \mathbb{R}^m)$. Furthermore, $J^0(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m$. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is smooth, then $j^0 f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$. Now we present the C^∞ Whitney topology following [GOLUB].

DEFINITION 2.12. *Let* X *and* Y *be smooth manifolds. Denote by* $C^\infty(X,Y)$, *the set of smooth mappings from* X *to* Y. *Choose a metric* d *on* $J^k(X,Y)$ *compatible with its topology. This is possible since all manifolds are metrizable. We now define a basis for the topology. For each smooth function* $f : X \longrightarrow Y$, *define* $B_\delta(f) = \{g \in C^\infty(X,Y) : \text{for all } x \in X, d(j^k f(x), j^k g(x)) < \delta(x)\}$ *where* $\delta : X \longrightarrow \mathbb{R}^+$ *is a continuous map. If we let these* $B_\delta(f)$ *range over all natural numbers* k, *all smooth functions* f, *and all continuous functions* δ, *then* $\{B_\delta(f)\}$ *is a basis for* $C^\infty(X,Y)$, *[GOLUB]. This is called the* C^∞ *Whitney topology on the set* $C^\infty(X,Y)$.

A helpful way to think about a particular basis element $\{B_\delta(f)\}$ is that it is the set of all smooth mappings of $X \to Y$, all of whose partial derivatives up to order k are δ-close to f's corresponding partial derivatives.

We require the next definition and proposition from [GOLUB] because we will soon use a theorem that says a particular subset of $C^\infty(X,Y)$ is residual. What we want to know is that this residual set is dense in the space $C^\infty(X,Y)$.

DEFINITION 2.13. *Let F be a topological space. Then a*

(I.) A subset G of F is residual if it is the countable intersection of open dense subsets of F.

(II.) F is a Baire space if every residual set is dense in F.

PROPOSITION 2.3. *Let X and Y be smooth manifolds. Then $C^\infty(X,Y)$ is a Baire space in the Whitney C^∞ topology.*

Proof: [GOLUB].

We also need this next proposition from [GOLUB].

PROPOSITION 2.4. *Let X, Y and Z be smooth manifolds. Then $C^\infty(X,Y)\times C^\infty(X,Z)$ is homeomorphic (in the Whitney C^∞ topology) with $C^\infty(X,Y\times Z)$ by using the standard identification $(f,g)\to f\times g$ where $(f\times g)(x)=(f(x),g(x))$.*

We now define multi-jets and state the Multijet Transversality Theorem from [GOLUB]. Multi-jets are essentially the "cartesian product" of jets.

DEFINITION 2.14. *Let X and Y be smooth manifolds. Define $X^s = X \times X \times \cdots \times X$ (s-times) and $X^{(s)} = \{(x_1, x_2, \ldots, x_s) \in X^s :$ for some i and j, $x_i \neq x_j\}$. Recall from definition* 2.11 *the source map $\alpha : J^k(X,Y) \longrightarrow X$. Define $\alpha^s : J^k(X,Y)^s \longrightarrow X^s$ as $\alpha^s(\sigma_1, \sigma_2, \ldots, \sigma_s) = (\alpha(\sigma_1), \alpha(\sigma_2), \ldots, \alpha(\sigma_s))$. Then we call ${J_s}^k(X,Y) = \alpha^{s-1}(X^{(s)})$ the s-fold k-jet bundle. $X^{(s)}$ is a manifold since it is an open subset of X^s. Thus, ${J_s}^k(X,Y)$ is an open subset of $J^k(X,Y)$ and is also a smooth manifold. Now let $f : X \longrightarrow Y$ be smooth. Then we define the multi-jet map of f as ${j_s}^k f : X^{(s)} \longrightarrow {J_s}^k(X,Y)$ where ${j_s}^k f(x_1, x_2, \ldots, x_s) = (j^k f(x_1), j^k f(x_2), \ldots, j^k f(x_s))$.*

For our purposes, we use the following type of multijets. Suppose f, g, r are smooth maps and $f, g, r : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Further, suppose $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{3n}$. where $\psi(x) = (f(x), g(x), r(x))$. Then for any $x, u, v \in \mathbb{R}^n$, we have ${j_3}^0(x,u,v) = (j^0\psi(x), j^0\psi(u), j^0\psi(v)) = (x, f(x), g(x), r(x), u, f(u), g(u), r(u), v, f(v), g(v), r(v))$.

THEOREM 2.3. *(Multijet Transversality Theorem). Let X and Y be smooth manifolds with W a submanifold of ${J_s}^k(X,Y)$. Let $T_W = \{f \in C^\infty(X,Y) : {j_s}^k f \pitchfork W\}$. Then T_W is a residual subset of $C^\infty(X,Y)$.*

Proof: [GOLUB].

Now that we have presented the definitions and theorems from [GOLUB], we consider the non-autonomous system $\{f, g, g, f, f, g, g, f, f, g, g, f, \ldots\}$. The following theorem says that any point of period 2 with respect to $\{f, g, g, f, f, g, g, f, f, g, g, f, \ldots\}$ is unstable. However, it is possible for a periodic orbit of period $4m$ to be stable with respect to $\{f, g, g, f, f, g, g, f, f, g, g, f \ldots\}$. As in the autonomous case,