Using topology and geometry to study the Henon attractor

The results from Section III suggest that more complex behavior is to be expected for non-autonomous dynamical systems. Consequently, in this section we turn our attention to a more complex attractor than a fixed point or periodic point.

Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as $T_{(a,b)}(x, y) = (1 + y - ax^2, bx)$ where $a = 1.4$ and $b = .3$. We prove that there is an open interval about the point $(1.4, 0.3) \in \mathbb{R}^2$ such that T has positive topological entropy. By [KATOK], in an open interval about the point $(1.4, .3)$, $T|_{(a,b)}$ has a transverse homoclinic point. We find a lower bound, $\log(1.272)$, for the topological entropy of the Henon map. We also show that the Henon attractor contains an infinite number of paths from each of four distinct path classes.

SKETCH OF PROOF

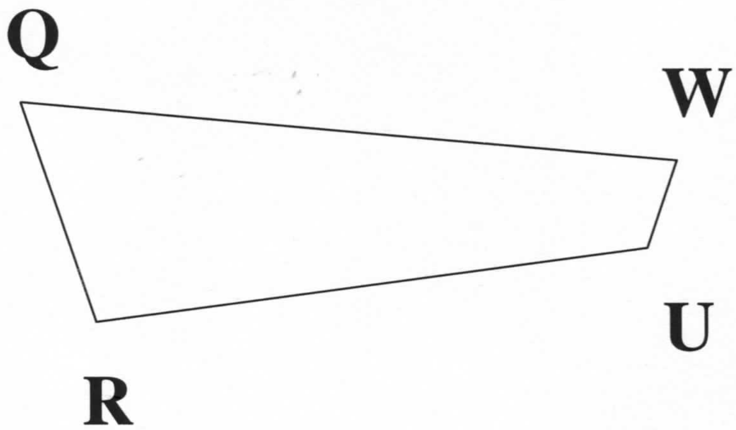
First, we define the standard quadrilateral $\mathfrak{Q}$ which acts as a trapping region of the attractor. Then we characterize the set $T^2(\mathfrak{Q})$. We define four different classes of paths lying in $T^2(\mathfrak{Q})$. These classes of paths are called $[\gamma], [\delta], [\beta]$, and $[\alpha]$. We show that these paths cover each other in the following sense: for any path $\sigma \in [\gamma]$, then $T \circ \sigma \in [\alpha]$, and $T \circ \sigma \in [\delta]$. From the path covers, we create a four by four matrix, called S.

Matrix powers, i.e. S^k, corresponds to function iteration $T^k(\sigma)$. We use this to show that the largest eigenvalue of S will be a lower bound on the growth rate of the number of distinct subpaths of $T^k(\sigma)$ lying in either $[\gamma], [\delta], [\beta]$, or $[\alpha]$. This allows us to show that there is a one dimensional disk D, such that the arc length of $T^k(D)$ grows at an exponential rate. In [YOMDIN], a relationship between the growth rate of a disk under iteration of the map and the entropy is established; they prove that the entropy of T is greater than or equal to this growth rate. This implies that the entropy of T is greater than log(1.272).

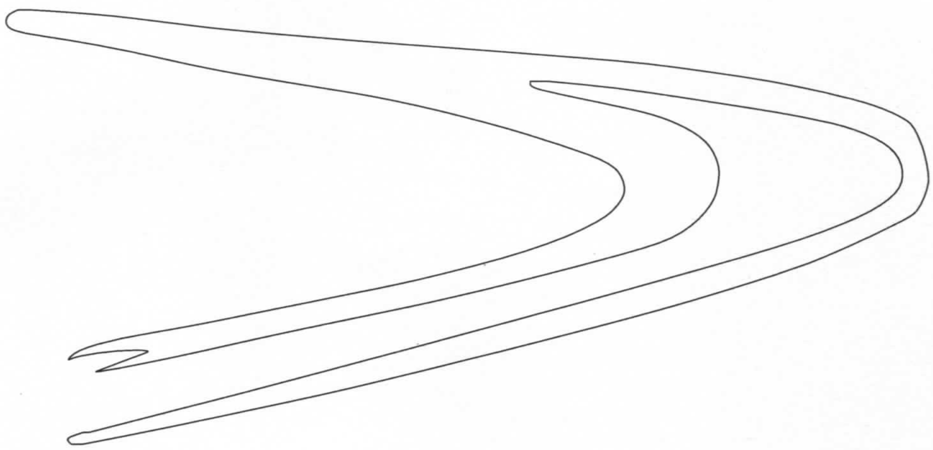
DISCUSSION

DEFINITION 5.22. *Set* $Q = (-1.33, .42)$*,* $R = (-1.06, -.5)$*,* $U = (1.245, -.14)$*, and* $W = (1.32, .133)$*. Set* $\mathfrak{Q} =$ *quadrilateral* $QRUW$*, and set* $\mathfrak{A} = T^2(\mathfrak{Q})$*.*

What follows is a picture of $\mathfrak{Q}$:



What follows is a picture of $\mathfrak{A}$:



Remark 5.24. *It is known that* $T(\mathfrak{Q}) \subset int(\mathfrak{Q})$. *(See [HENON].) Hence,* $\mathfrak{Q}$ *is a trapping region.*

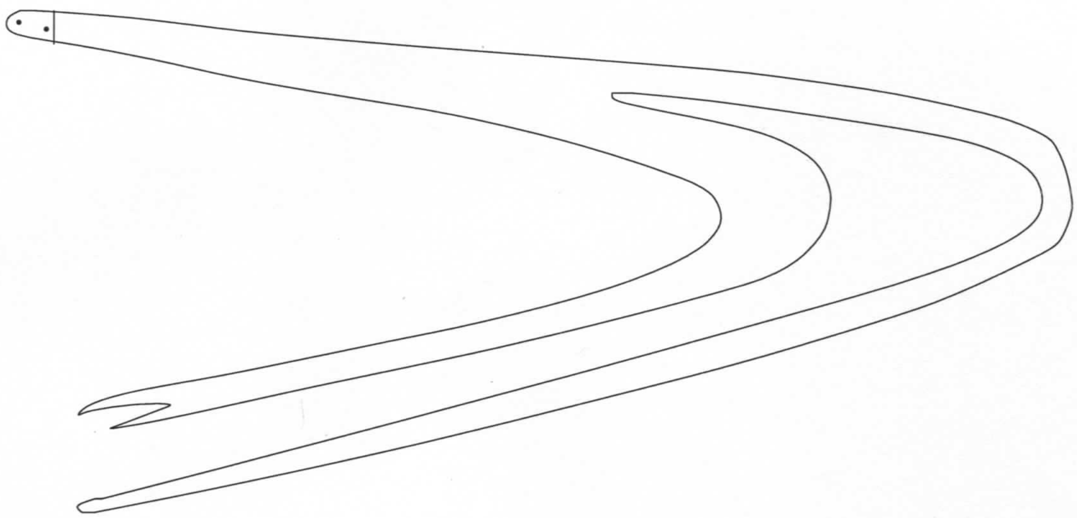
Corollary 5.6. $\mathfrak{A}$ *is an invariant set i.e.* $T^k(\mathfrak{A}) \subset \mathfrak{A}$ *for all* $k \geq 1$.

Proof: Claim: $T(\mathfrak{A}) \subset \mathfrak{A}$.
By Remark 5.6, $T(\mathfrak{Q}) \subset int(\mathfrak{Q})$, so $\mathfrak{A} \subset T(int(\mathfrak{Q})) \subset T(\mathfrak{Q})$. Apply T to both sides; we obtain $T(\mathfrak{A}) \subset \mathfrak{A}$. Now we apply induction and use the fact that $T(\mathfrak{A}) \subset \mathfrak{A}$. ■

Next we divide the set $\mathfrak{A}$ into regions so that we can define classes of paths in $\mathfrak{A}$.

Definition 5.23. *Set Region* $A = \{(x, y) \in \mathfrak{A} : x \leq -1.1\}$.

Region A



The following may seem a bit strange, but a convenient way to define the classes of paths is to define predicates, and then use these predicates to define particular regions of $\mathfrak{A}$.

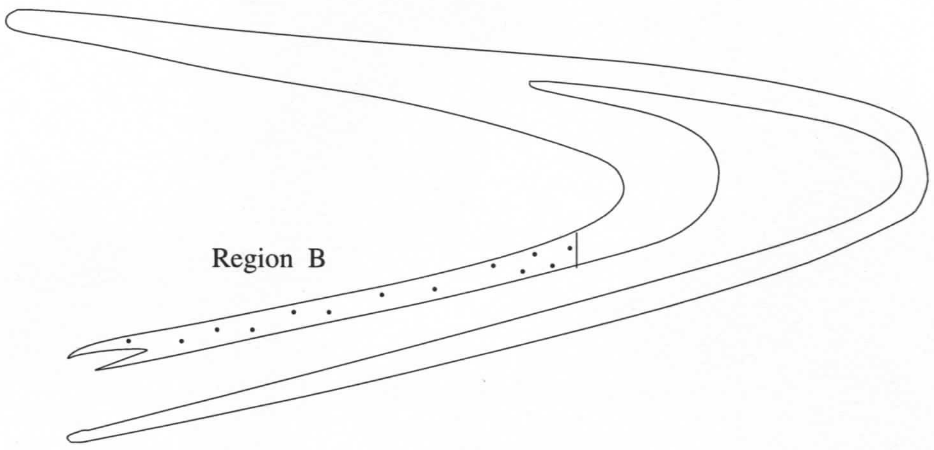
Definition 5.24. *Define the predicate* $P_B : \mathfrak{A} \longrightarrow \{$ *True, False* $\}$ *as follows: Set* $P_B(x, y) =$ *True, if there is a path* $\mu : [0, 1] \longrightarrow \mathfrak{A}$ *satisfying conditions I through III.*

I. $\mu(t) = (\mu_x(t), \mu_y(t))$,

II. $\mu(0) = (x, y)$ *and* $\mu(1) = (0.56, y_0)$, *where* $(0.56, y_0) \in (\{0.56\} \times [-.05, -.15])$ $\cap\ \mathfrak{A}$,

III. $\mu_x(t) \leq 0.56$ *for all* $t \in [0, 1]$.

Otherwise, set $P_B(x, y) =$*False. Set Region* $B = \{(x, y) \in \mathfrak{A} : P_B(x, y) =$ *True*$\}$.



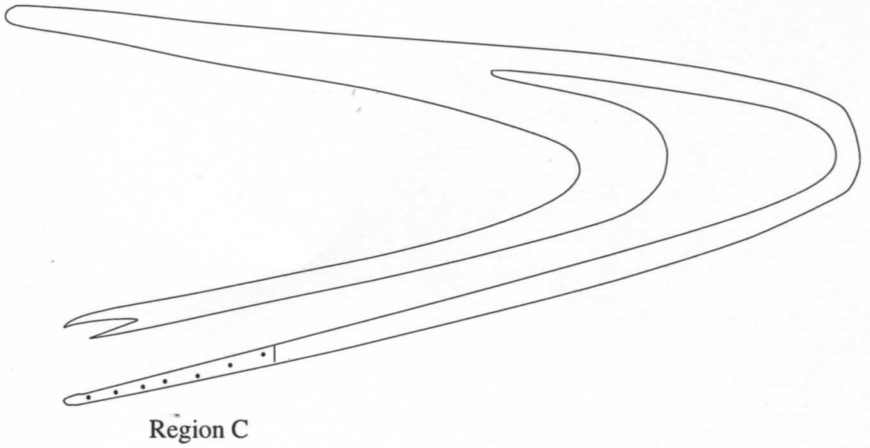
DEFINITION 5.25. *Define the predicate* $P_C : \mathfrak{A} \longrightarrow$ {*True, False* } *as follows: Set* $P_C(x,y) =$ *True, if there is a path* $\mu : [0,1] \longrightarrow \mathfrak{A}$ *satisfying conditions I through III.*

I. $\mu(t) = (\mu_x(t), \mu_y(t))$,

II. $\mu(0) = (x,y)$, *and* $\mu_x(1) = -0.31$,

III. $\mu_y(1) \leq -.30$ *and* $\mu_x(t) \leq -0.31$ *for all* $t \in [0,1]$.

Otherwise, set $P_C(x,y) =$*False. Set Region* $C = \{(x,y) \in \mathfrak{A} : P_C(x,y) = True\}$.



Definition 5.26. *Define the predicate* $P_D : \mathfrak{A} \longrightarrow \{True,\ False\}$ *as follows: Set* $P_D(x, y) =$ *True, if there is a path* $\mu : [0, 1] \longrightarrow \mathfrak{A}$ *satisfying conditions I through IV.*

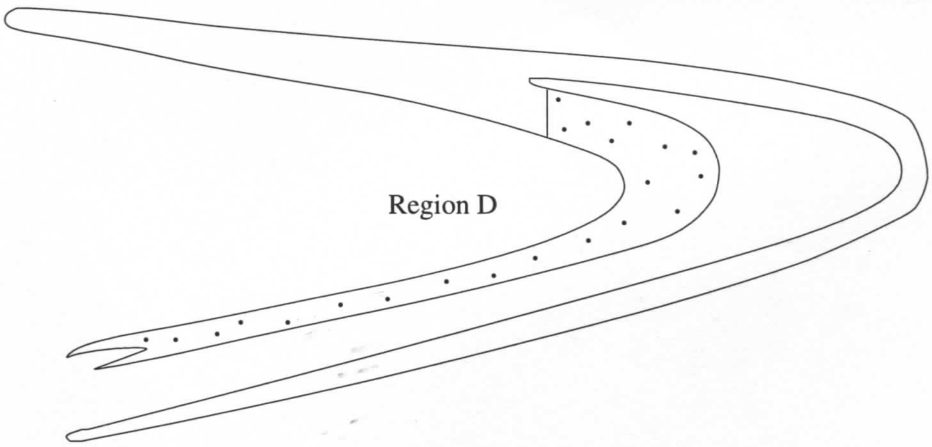
I. $\mu(t) = (\mu_x(t), \mu_y(t))$,

II. $\mu(0) = (x, y)$ *and* $\mu(1) = (0.4, 0.17)$,

III. For all $t \in [0, 1]$, *if* $\mu_x(t) < 0.4$, *then* $\mu_y(t) \leq -0.1$,

IV. For all $t \in [0, 1]$, *if* $\mu_x(t) \geq 0.4$, *then* $\mu_y(t) \leq 0.195$.

Otherwise, $P_D(x, y) =$*False. Set Region* $D = \{(x, y) \in \mathfrak{A} : P_D(x, y) = True\}$.

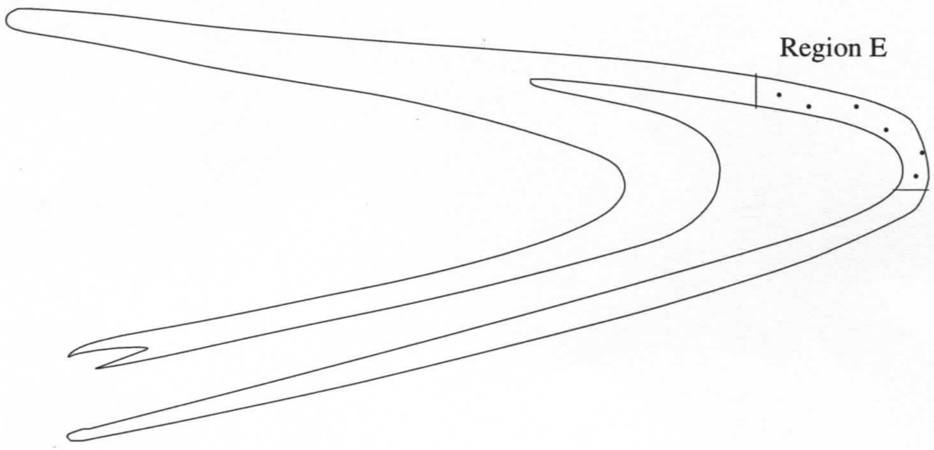


Definition 5.27. *Define the predicate* $P_E : \mathfrak{A} \longrightarrow \{True,\ False\}$ *as follows: Set* $P_E(x, y) =$ *True, if there is a path* $\mu : [0, 1] \longrightarrow \mathfrak{A}$ *satisfying condition I through IV.*

I. $\mu(t) = (\mu_x(t), \mu_y(t))$,

II. $\mu(0) = (x, y)$ *and* $\mu(1) = (1.26, -0.02)$, *III.* For all $t \in [0,1]$, $\mu_x(t) \geq 0.9$ and $\mu_y(t) \geq -0.02$

Otherwise, set $P_E(x,y)$ =*False. Set Region* $E = \{(x,y) \in \mathfrak{A} : P_E(x,y) = \textit{True}\}$.



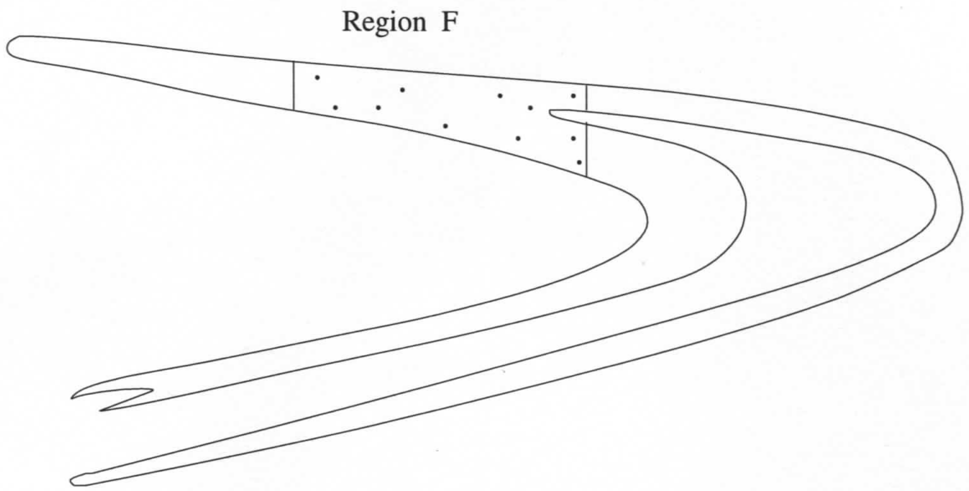
DEFINITION 5.28. *Define the predicate* $P_F : \mathfrak{A} \longrightarrow$ { *True, False* } *as follows: Set* $P_F(x,y) =$ *True, if there is a path* $\mu : [0,1] \longrightarrow \mathfrak{A}$ *satisfying condition I thru III.*

I. $\mu(t) = (\mu_x(t), \mu_y(t))$.

II. $\mu(0) = (x,y)$ *and* $\mu(1) = (0.0, 0.25)$.

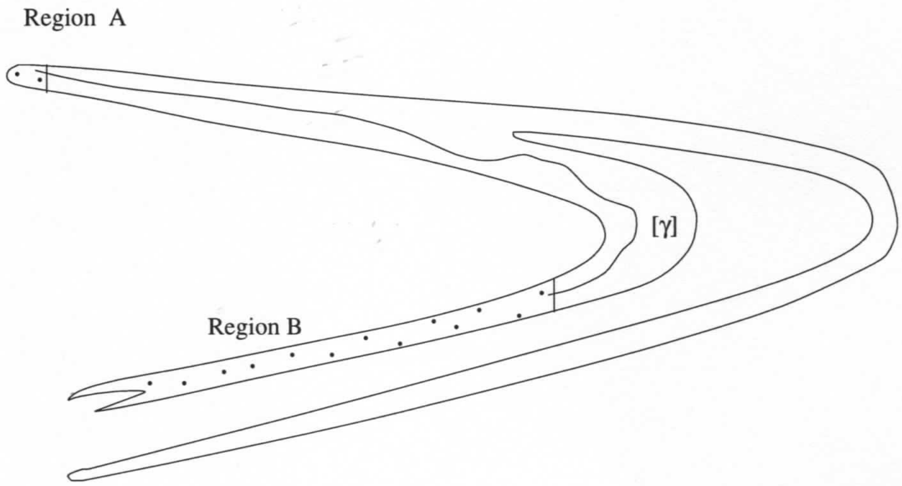
III. For all $t \in [0,1]$, *we have* $-0.5 \leq \mu_x(t) \leq 0.5$ *and* $\mu_y(t) \geq 0.1$.

Otherwise, set $P_F(x,y)$ =*False. Set Region* $F = \{(x,y) \in \mathfrak{A} : P_E(x,y) = \textit{True}\}$.

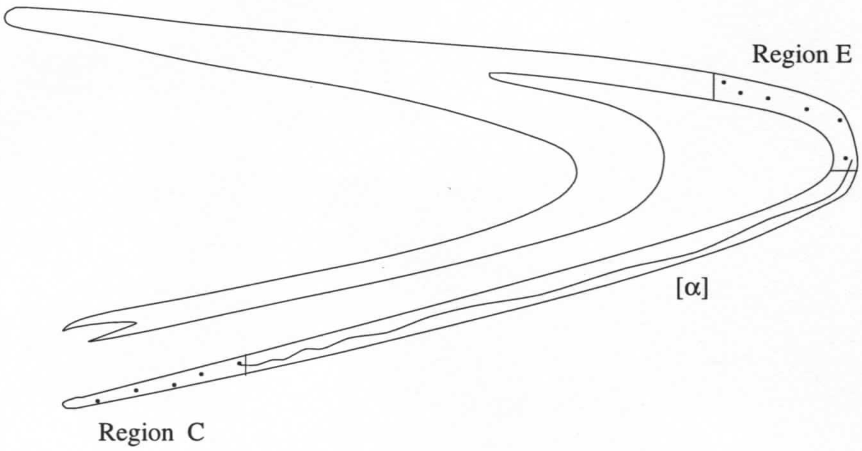


Now, we define the four path classes, and show pictures of a typical path. The pictures enable us to see the geometry and topology in the proof.

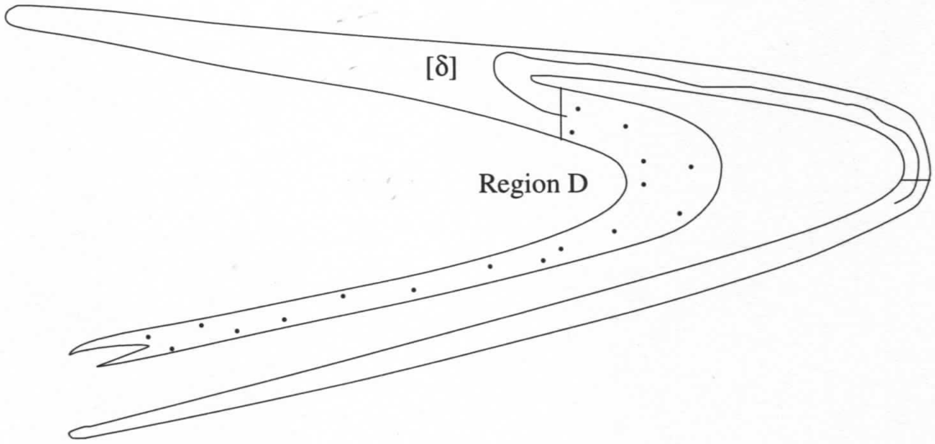
DEFINITION 5.29. *Define* $[\gamma] = \{\mu : [a,b] \longrightarrow \mathfrak{A} |\ 0 \leq a < b \leq 1$ *and* μ *is a* C^∞ *path and there exists* $s, t \in [a,b]$ *such that* $\mu(s) \in$ *Region A and* $\mu(t) \in$ *Region B*$\}$



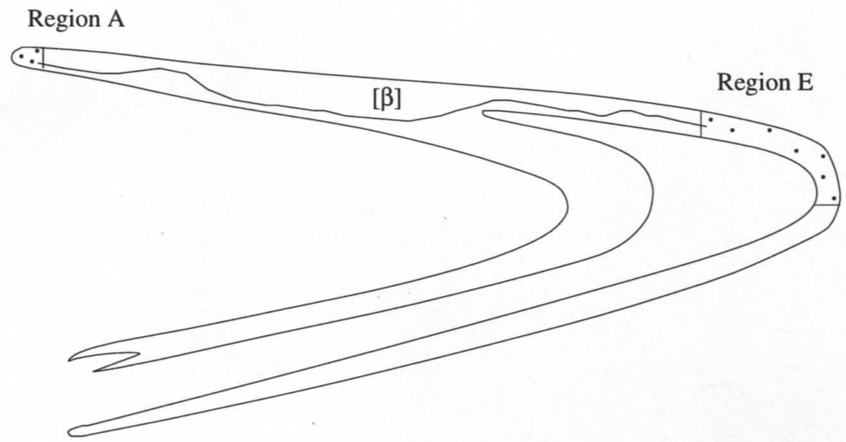
DEFINITION 5.30. *Define* $[\alpha] = \{\mu : [a,b] \longrightarrow \mathfrak{A} |\ 0 \leq a < b \leq 1$ *and* μ *is a* C^∞ *path and there exists* $s, t \in [a,b]$ *such that* $\mu(s) \in$ *Region E and* $\mu(t) \in$ *Region C*$\}$



DEFINITION 5.31. *Define* $[\delta] = \{\mu : [a,b] \longrightarrow \mathfrak{A} |\ 0 \leq a < b \leq 1$ *and* μ *is a* C^∞ *path and there exists* $s, t \in [a,b]$ *such that* $\mu(s) \in$ *Region* $E \cap [1.22, 1.28] \times \{-0.02\}$ *and* $\mu(t) \in$ *Region D*$\}$



Definition 5.32. *Define* $[\beta] = \{\mu : [a,b] \longrightarrow \mathfrak{A} |\ 0 \leq a < b \leq 1$ *and* μ *is a* C^∞ *path and there exists* $s, t \in [a,b]$ *such that* $\mu(s) \in$ *Region A and* $\mu(t) \in$ *Region E*$\}$



In this next section, we define what it means for one path class to T cover the other path class. We rely on this notion to construct a lower bound on the length of the path, $T^k(\sigma)$ as we iterate T. First, we require technical Lemmas 5.16, 5.17, ??, 5.18, 5.19, 5.20, and 5.21 to prove that certain path classes cover other path classes.

Lemma 5.16. $T[(x = -1.1) \cap$ *Region A*$] \subset$ *Region C*.

Proof: See the appendix.

Lemma 5.17. $T[(x = .56) \cap$ *Region B*$] \subset$ *Region D*.

Proof: See the appendix.

LEMMA 5.18. *$T([1.22, 1.28] \times \{-0.02\} \ \cap$ Region $E) \subset$ Region A.*

Proof: See the appendix.

LEMMA 5.19. *$T((x = -.31) \ \cap$ Region $C) \subset$ Region B.*

Proof: See the appendix.

LEMMA 5.20. *$T((x = 0.9) \ \cap$ Region $E) \subset$ Region F.*

Proof: See the appendix.

LEMMA 5.21. *$T((x = 0.40) \ \cap$ Region $D) \subset$ Region E.*

Proof: See the appendix.

DEFINITION 5.33. *Suppose $[\mu]$ and $[\nu]$ are one of the four path classes defined above. We say $[\mu]$ T path covers $[\nu]$ if for any $\theta \in [\mu]$, then $T \circ \theta \in [\nu]$.*

DEFINITION 5.34. *Let $\mu : [a, b] \longrightarrow \mathfrak{A}$ be a path. A path ν is a subpath of μ if $\nu : [q, s] \longrightarrow \mathfrak{A}$ and $a \leq q < s \leq b$, where $\mu(t) = \nu(t)$ for all $t \in [q, s]$.*

DEFINITION 5.35. *$\nu_1, \nu_2 : [a,b] \longrightarrow \mathfrak{A}$ are distinct subpaths of μ if there exist q, r_1, r_2, s, where $q < r_1 \leq r_2 < s$, satisfying either $\nu_1(t) = \mu|_{[q,r_1]}$ and $\nu_2(t) = \mu|_{[r_2,s]}$ OR $\nu_2(t) = \mu|_{[q,r_1]}$ and $\nu_1(t) = \mu|_{[r_2,s]}$.*

Define the following subpath matrix:

$$S = \begin{pmatrix} & \gamma & \alpha & \delta & \beta \\ \gamma & 0 & 1 & 1 & 0 \\ \alpha & 1 & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 & 1 \\ \beta & 0 & 1 & 0 & 0 \end{pmatrix}$$

REMARK 5.25. *We refer to entries of the matrix S as follows: For example, $[S]_{\gamma\gamma} = 0$, and $[S]_{\beta\alpha} = 1$. When we multiply S times itself k times, we use $[S^k]_{\beta\alpha}$ to denote the $\beta\alpha$ entry of the matrix S^k.*

Suppose ν is a path in the path class $[\lambda]$. This next Lemma states that the $\lambda\theta$ entry of the matrix, S, records whether the path $T(\nu)$ has a subpath that lies in the path class $[\theta]$.

LEMMA 5.22. *Suppose $\lambda, \theta \in \{\gamma, \alpha, \delta, \beta\}$. Suppose $\nu : [a,b] \longrightarrow \mathfrak{A}$ is a path such that $\nu \in [\lambda]$. If $[S]_{\lambda\theta} = 1$, then $T \circ \nu \in [\theta]$.*

Proof: (By brute force.)

Case (I.) Suppose $\nu \in [\gamma]$. Notice that $[S]_{\gamma\alpha} = 1$, and $[S]_{\gamma\delta} = 1$. By the definition of $[\gamma]$, there are $s_1 \neq s_2$ (W.L.O.G. assume $s_1 < s_2$) and $\nu(s_1) \in$ Region A and $\nu(s_2) \in$ Region B. By the definition of Region A and Region B and the fact that ν is a path, there exist real numbers t_1, t_2 satisfying $s_2 \leq t_1 < t_2 \leq s_2$ so that $\nu(t_1) \in$ Region $A \cap \ (x = -1.1)$ and $\nu(t_2) \in$ Region $B \cap (x = .56)$.

By Lemma 5.16, $T \circ \nu(t_1) \in$ Region C, and by Lemma 5.17, $T \circ \nu(t_2) \in$ Region D. Since $T \circ \nu|_{[t_1,t_2]}$ is a path between Regions C and D and because of the topology of $\mathfrak{A}$, there exists $s \in [t_1, t_2]$ such that $T \circ \nu(s) \in [1.22, 1.28] \times \{-0.02\} \ \cap$ Region E. Hence, $T \circ \nu|_{[t_1,s]} \in [\alpha]$ and $T \circ \nu|_{[s,t_2]} \in [\delta]$.

Case (II.) Suppose $\nu \in [\alpha]$. Thus, $[S]_{\alpha\gamma} = 1$. By the definition of $[\alpha]$ (using similar reasoning as for $[\gamma]$) there exists $t_3 \neq t_4$ (W.L.O.G. assume $t_3 < t_4$) such that $\nu(t_3) \in$ Region $E \cap (y = -0.02)$ and $\nu(t_4) \in$ Region $C \cap (x = -0.31)$. Because of Lemma 5.18 and the fact that Region $E \cap (y = -0.02) \subset [1.22, 1.28] \times \{-0.02\}$, (see appendix), $T \circ \nu(t_3) \in$ Region A; and because of Lemma 5.19, $T \circ \nu(t_4) \in$ Region B. Hence, $T \circ \nu|_{[t_3,t_4]} \in [\gamma]$.

Case (III.) Suppose $\nu \in [\delta]$. Thus, $[S]_{\delta\beta} = 1$. By the definition, of $[\delta]$, there exists $t_5 \neq t_6$ (W.L.O.G. assume $t_5 < t_6$) such that $\nu(t_5) \in$ Region $E \cap [1.22, 1.28] \times \{-0.02\}$ and $\nu(t_6) \in$ Region $D \cap (x = .4)$. By Lemma 5.18, $T \circ \nu(t_5) \in$ Region A. By Lemma 5.21, $T \circ \nu(t_6) \in$ Region E. Hence, $T \circ \nu|_{[t_5,t_6]} \in [\beta]$.

Case (IV.) Suppose $\nu \in [\beta]$. Now, $[S]_{\beta\alpha} = 1$. By the definition, of $[\beta]$, there exists $t_7 \neq t_8$ (W.L.O.G. assume $t_7 < t_8$) such that $\nu(t_7) \in$ Region $A \cap (x = -1.1)$ and $\nu(t_8) \in$ Region $E \cap (x = .9)$. By Lemma 5.16, $T \circ \nu(t_7) \in$ Region C. By Lemma 5.20, $T \circ \nu(t_8) \in$ Region F. Since $T \circ \nu|_{[t_7,t_8]}$ is a path between Region C and Region F and because of the topology of $\mathfrak{A}$, there exists $s \in [t_7, t_8]$ such that $T \circ \nu(s) \in$ Region E. Hence, $T \circ \nu|_{[t_7,s]} \in [\alpha]$. ■

We now define a particular path μ in $\mathfrak{A}$ so that the length of the path $T^k(\mu)$ grows exponentially as we increase k.

DEFINITION 5.36. *Define* $\mu : [0,1] \longrightarrow \mathfrak{A}$ *so that* μ *is* C^∞ *and* μ *has at least* 4 *distinct subpaths so that one subpath lies in* $[\gamma]$*, one subpath lies in* $[\alpha]$*, one subpath lies in* $[\delta]$*, and one subpath lies in* $[\beta]$*.*

DEFINITION 5.37. *Suppose* $M = (m_{ij})$ *is an* $n \times n$ *matrix. Set* $\|M\| = \sum_{j=1}^{n}\sum_{i=1}^{n} m_{ij}$.

This next lemma proves that the sum of all entries in the matrix S^k counts the number of distinct subpaths of $T^k(\mu)$ that lie in one of the path classes, $[\gamma]$, $[\alpha]$, $[\delta]$, or $[\beta]$.

LEMMA 5.23. *We can find a set of distinct subpaths of* $T^k \circ \mu$ *so that each subpath lies in either* $[\gamma]$*,* $[\alpha]$*,* $[\delta]$*, or* $[\beta]$*. Further, the number of elements in this set (the number of distinct subpaths) is* $\geq \|S^k\|$*.*

Proof: Consider $T^k \circ \mu$. We use induction on k.

Base Case: $k = 1$.

By the hypothesis, there exists 4 intervals: $[t_1, t_2]$, $[t_3, t_4]$, $[t_5, t_6]$, $[t_7, t_8]$ that are pairwise disjoint or pairwise intersect in exactly one point, and $\mu|_{[t_1,t_2]} \in [\gamma]$, $\mu|_{[t_3,t_4]} \in [\alpha]$, $\mu|_{[t_5,t_6]} \in [\delta]$, and $\mu|_{[t_7,t_8]} \in [\beta]$. By Lemma 5.22, there are distinct subpaths $T \circ \mu|_{[t_1,s_1]} \in [\alpha]$, $T \circ \mu|_{[s_2,t_2]} \in [\delta]$, and $s_1 \leq s_2$ OR $T \circ \mu|_{[t_1,s_1]} \in [\delta]$, $T \circ \mu|_{[s_2,t_2]} \in [\alpha]$, and $s_1 \leq s_2$. Since $S_{\alpha\gamma} = 1$, $S_{\delta\beta} = 1$, $S_{\beta\alpha} = 1$, by Lemma 5.22,

$T \circ \mu|_{[t_3,t_4]} \in [\gamma]$, $T \circ \mu|_{[t_5,t_6]} \in [\beta]$, and $T \circ \mu|_{[t_7,t_8]} \in [\alpha]$. Further, all 5 of these subpaths of $T \circ \mu$ are distinct. The proof for the Base Case is complete.

The induction hypothesis is long, so we break it into two parts, (I.) and (II.). For each $\epsilon, \theta, \lambda \in \{\gamma, \alpha, \delta, \beta\}$,

(I.) Suppose that $T^k \circ \mu$ has $[S^k]_{\epsilon\lambda}$ distinct subpaths which are elements of $[\lambda]$, and all of these subpaths are of the form $T^k \circ \mu|_{[s,t]}$ where $\mu|_{[t_i,t_{i+1}]} \in [\epsilon]$ and $i \in \{1, 3, 5, 7\}$, and $t_i \leq s < t \leq t_{i+1}$.

(II.) Suppose any $\lambda, \lambda' \in \{\gamma, \alpha, \delta, \beta\}$, where $\lambda \neq \lambda'$, satisfy the following: if $T^k \circ \mu|_{[s,t]} \in [\lambda]$ is one of these $[S^k]_{\epsilon\lambda}$ paths and if $T^k \circ \mu|_{[u,v]}$ is one of these $[S^k]_{\epsilon\lambda'}$ paths then either $t \leq u$ or $v \leq s$ i.e. $[s,t] \cap [u,v] = \emptyset$ or $[s,t] \cap [u,v] =$ a single point.

Notice that the base case trivially satisfies parts (I.) and (II.). Further, each distinct subpath, $T^k \circ \mu|_{[s,t]}$, lying in the set of all $[S^k]_{\epsilon\lambda}$ paths in $[\lambda]$, satisfies the following: if $S_{\lambda\theta} = 1$, then by Lemma 5.22, $T \circ T^k \circ \mu|_{[s,t]} \in [\theta]$.

REMARK 5.26. *We need to be sure that for any two distinct subpaths* $T^k \circ \mu|_{[u_1,v_1]}, T^k \circ \mu|_{[u_2,v_2]} \in [\lambda]$, *where* $t_i \leq u_1 < v_1 \leq u_2 < v_2 \leq t_{i+1}$, *then* $T \circ T^k \circ \mu|_{[u_1,v_1]}$ *and* $T \circ T^k \circ \mu|_{[u_2,v_2]}$ *are distinct subpaths of* $T^{k+1} \circ \mu$. *This is trivial because* $v_1 \leq v_2$ *so this implies* $T^{k+1} \circ \mu|_{[u_1,v_1]}$, $T^{k+1} \circ \mu|_{[u_2,v_2]}$ *are distinct subpaths of* $T^{k+1} \circ \mu$.

(III.) Further, if $T^k \circ \mu|_{[u_1,v_1]} \in [\lambda]$ and $T^k \circ \mu|_{[u_2,v_2]} \in [\lambda']$ and $\lambda \neq \lambda'$, then by (II.) $[u_1, v_1] \cap [u_2, v_2] = \emptyset$ or $[u_1, v_1] \cap [u_2, v_2] =$ a single point. Thus, for any two paths, where one path is a subpath of $T^{k+1} \circ \mu|_{[u_1,v_1]}$ and the other is a subpath of $T^{k+1} \circ \mu|_{[u_2,v_2]}$, are distinct.

By (I.), (II.), and (III.), there are $[S^k]_{\epsilon\gamma}[S]_{\gamma\theta} + [S^k]_{\epsilon\alpha}[S]_{\alpha\theta} + [S^k]_{\epsilon\delta}[S]_{\delta\theta} + [S^k]_{\epsilon\beta}[S]_{\beta\theta}$ distinct subpaths of $T^{k+1} \circ \mu \in [\theta]$. This sum equals $[S^{k+1}]_{\epsilon\theta}$ by the definition of matrix multiplication. We also see by Remark 5.26 and (III.) that the distinct paths of $T^{k+1} \circ \mu$ corresponding to $[S^{k+1}]$ satisfy the induction hypotheses.

■

REMARK 5.27. *The spectral radius, $\sigma(S)$ of the path matrix, S, is greater than 1. Hence,* $\log \sigma(S) \geq \log(1.272)$.

Proof: We calculate the characteristic polynomial $p(\lambda)$ of S,

$$p(\lambda) = \begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 1 & 0 & -\lambda \end{vmatrix}$$

Expanding along the first column we obtain:

$$p(\lambda) = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} +$$

$$-1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= (-\lambda)(-\lambda)\lambda^2 + (-1)[\lambda^2 - 1(-1)]$$

$$= \lambda^4 - \lambda^2 - 1.$$

Thus, $p(2) = 16 - 4 - 1 > 0$ and $p(1.272) < 0$, so $p(\lambda)$ has a root greater than 1.272. ■

REMARK 5.28. *Suppose $d(x, y)$ is the Euclidean metric in $\mathbb{R}^2$. If Q and P are subsets of $\mathbb{R}^2$, set $\rho(Q, P) = \inf\{d(q, p) : q \in Q$ and $p \in P\}$. Set $r = \min\{\rho($ Region A, Region B$), \rho($ Region E, Region C$), \rho($ Region A, Region E$), \rho($ Region E, Region D$)\}$. Then $r > 0$ because all the regions are compact and pairwise disjoint. Thus, we see that for any path $\nu \in [\gamma] \cup [\alpha] \cup [\delta] \cup [\beta]$ implies that the length of the path $\nu \geq r$.*

LEMMA 5.24. *The length of the path $T^k \circ \mu \geq \|S^k\| r$.*

Proof: (Notation: if σ is a path, we write $\mathfrak{L}(\sigma)$ to represent the arclength of σ.) Set $N = \|S^k\|$. Since S is a matrix of 0's and 1's, N is a natural number. By Lemma 5.23, we see that $T^k \circ \mu$ has N distinct subpaths each of which is in $[\gamma] \cup [\alpha] \cup [\delta] \cup [\beta]$. Hence, we have intervals $[s_1, t_1], \ldots, [s_N, t_N]$ that are pairwise disjoint or intersect pairwise in a single point, and $T^k \circ \mu|_{[s_i, t_i]} \in [\gamma] \cup [\alpha] \cup [\delta] \cup [\beta]$.

Now for any path $\theta : [a, b] \longrightarrow \mathfrak{A}$ and any s, u, v, t where $a \leq s < u \leq v < t \leq b$, then the paths satisfy the inequality: $\mathfrak{L}(\theta|_{[s,t]}) \geq \mathfrak{L}(\theta|_{[s,u]}) + \mathfrak{L}(\theta|_{[v,t]})$. This implies that $\mathfrak{L}(T^k \circ \mu) \geq \sum_{i=1}^{N} \mathfrak{L}(T^k \circ \mu|_{[s_i, t_i]}) \geq \sum_{i=1}^{N} r = \|S^k\| r$. ■

LEMMA 5.25. *The lower bound is* $\log(\sigma(S))$, *where* $\sigma(S)$ *is the spectral radius of* S,

$$\limsup_{k\to\infty}\frac{\log[\|S^k\|r]}{k}\geq\log\sigma(S).$$

Proof:

Since $r>0$ and r is independent of k,

$$\limsup_{k\to\infty}\frac{\log[\|S^k\|r]}{k}=\limsup_{k\to\infty}\frac{\log\|S^k\|+\log r}{k}=\limsup_{k\to\infty}\frac{\log\|S^k\|}{k}.$$

Also, we have $\limsup_{k\to\infty}\frac{\log\|S^k\|}{k}\geq\limsup_{k\to\infty}\frac{1}{k}\log(\text{ trace } S^k)$. We finish the proof by showing that $\limsup_{k\to\infty}\frac{1}{k}\log(\text{ trace } S^k)=\log\sigma(S)$. In [MISIUR], they prove that $\limsup_{k\to\infty}[trace(S^k)]^{\frac{1}{k}}=\sigma(S)$. Since the logarithm function is continuous and increasing, $\log\limsup_{k\to\infty}[trace(S^k)]^{\frac{1}{k}}=\limsup_{k\to\infty}\log[trace(S^k)]^{\frac{1}{k}}=\limsup_{k\to\infty}\frac{1}{k}\log[trace(S^k)]$. Hence, $\log\sigma(S)=\limsup_{k\to\infty}\frac{1}{k}\log[trace(S^k)]$. ■

THEOREM 5.21. *The entropy of the Henon map with parameters* $a=1.4$ *and* $b=.3$, *is bounded below by* $\log(1.272)$, $h(T)\geq\log(1.272)$.

Proof: Let $\|T^k\circ\mu\|$ denote the arc length of the curve $T^k\circ\mu$. From [YOMDIN],

$$h(T)\geq\limsup_{k\to\infty}\frac{1}{k}\log\|T^k\circ\mu\|$$

$$\geq\limsup_{k\to\infty}\frac{\log[\|S^k\|r]}{k}$$

by Lemma 5.24. The previous expression is

$$\geq\log\sigma(S)$$

by Lemma 5.25. The previous expression is

$$\geq \log(1.272)$$

by Lemma 5.27. ■

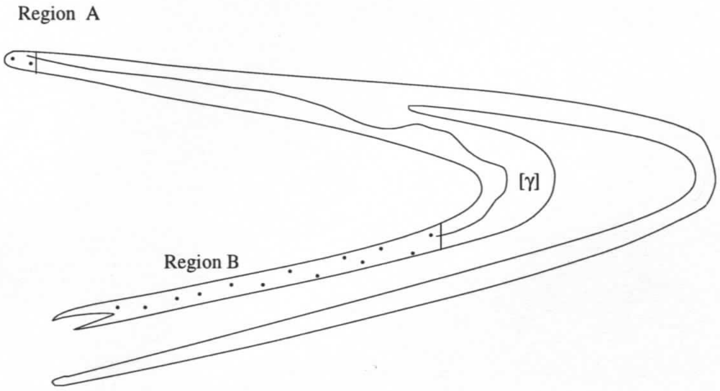
THEOREM 5.22. *If f is a $C^{1+\alpha}$, $(\alpha > 0)$, diffeomorphism of a compact two-dimensional manifold $h(f) > 0$, then f has a hyperbolic periodic point with a transversal homoclinic point.*

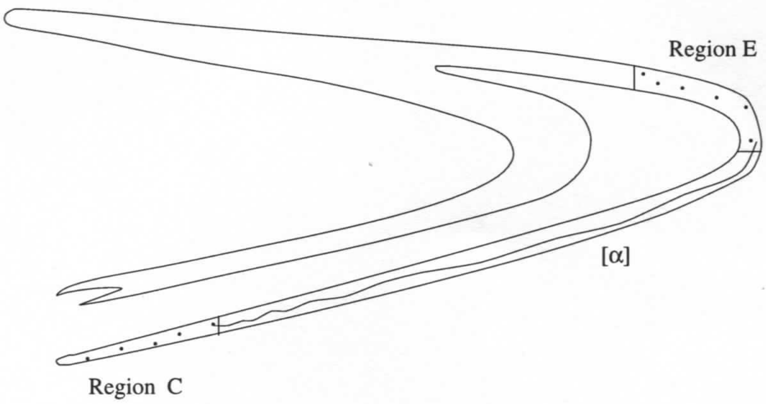
Proof: [KATOK].

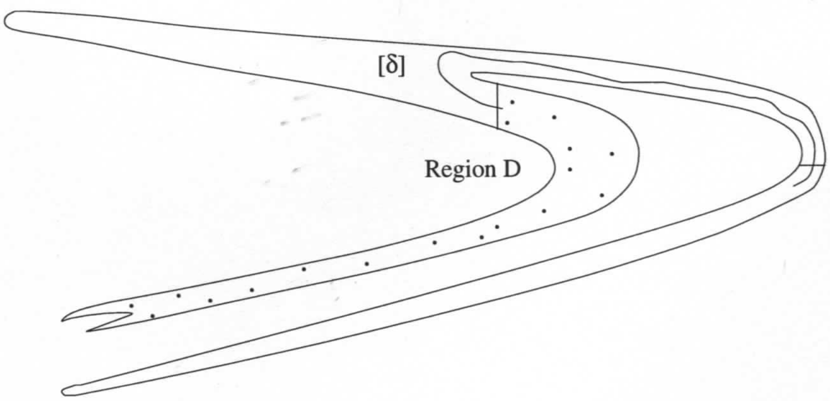
COROLLARY 5.7. *For some open neighborhood about the point $(1.4, 0.3)$, the Henon map $T_{(a,b)}(x, y) = (1 + y - ax^2, bx)$ has a transverse homoclinic point.*

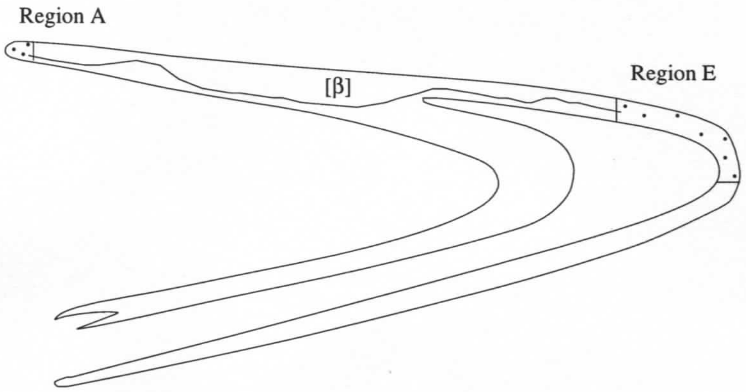
Proof: Since $T_{(a,b)}$ is a C^∞ diffeomorphism, and $h(T) > 0$, Theorem 5.22 implies that $T_{(1.4,0.3)}$, has a transverse homoclinic point. ■

COROLLARY 5.8. *The Henon attractor ($A = 1.4$, $B = 0.3$) has the following geometry. Consider the following four path classes defined earlier:*


Region A
[γ]
Region B


Region E
[α]
Region C


[δ]
Region D



The Henon attractor, $\bigcap_{k=1}^{\infty} T^k(\Omega)$*, contains an infinite number of copies of each path class* γ, δ, β*, and* α*.*

Proof: In the proof of Lemma 5.23, we show that for any path $\sigma \in [\gamma] \cup [\delta] \cup [\beta] \cup [\alpha]$, and for any large $m \in \mathbb{N}$ that we can find a large enough k such that $T^k(\sigma)$ contains more than m subpaths in $[\gamma]$, more than m subpaths in $[\delta]$, more than m subpaths in $[\beta]$, and more than m subpaths in $[\alpha]$. Since m is arbitrary and $T^2(\Omega)$ contains all four classes of paths, then the Henon attractor, $\bigcap_{k=1}^{\infty} T^k(\Omega)$ contains an infinite number of paths of each class: $[\gamma], [\delta], [\beta]$, and $[\alpha]$. ■

APPENDIX

The goal here is to characterize the geometry of $T^2(\Omega)$. As a reminder, Ω is the quadrilateral $QRUW$, which is a trapping region: