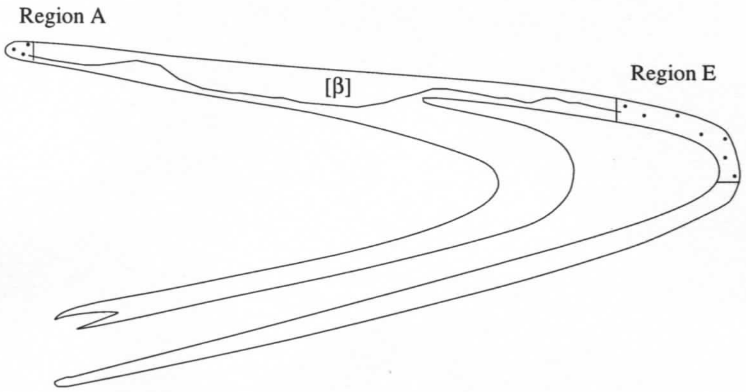
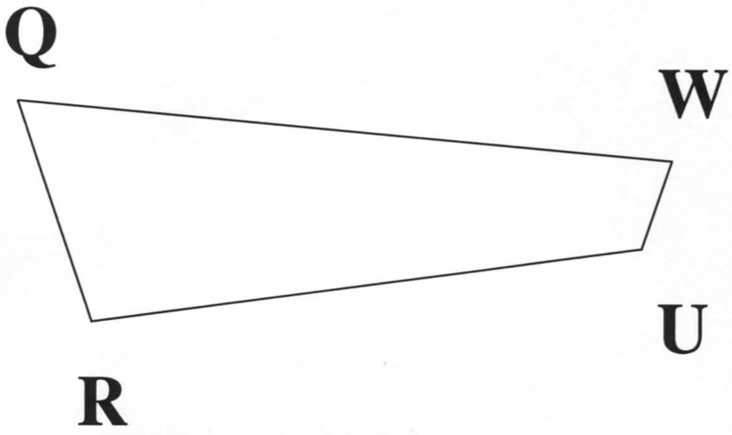


The Henon attractor, $\bigcap_{k=1}^{\infty} T^k(\Omega)$*, contains an infinite number of copies of each path class* γ, δ, β*, and* α*.*

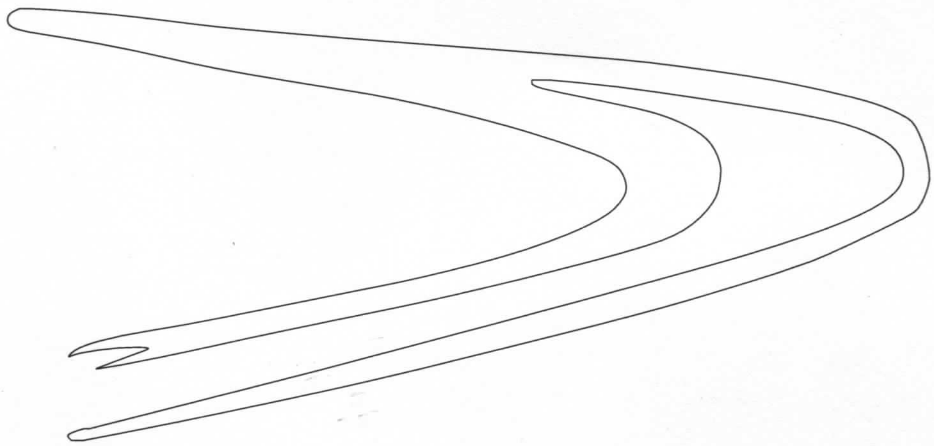
Proof: In the proof of Lemma 5.23, we show that for any path $\sigma \in [\gamma] \cup [\delta] \cup [\beta] \cup [\alpha]$, and for any large $m \in \mathbb{N}$ that we can find a large enough k such that $T^k(\sigma)$ contains more than m subpaths in $[\gamma]$, more than m subpaths in $[\delta]$, more than m subpaths in $[\beta]$, and more than m subpaths in $[\alpha]$. Since m is arbitrary and $T^2(\Omega)$ contains all four classes of paths, then the Henon attractor, $\bigcap_{k=1}^{\infty} T^k(\Omega)$ contains an infinite number of paths of each class: $[\gamma], [\delta], [\beta]$, and $[\alpha]$. ■

APPENDIX

The goal here is to characterize the geometry of $T^2(\Omega)$. As a reminder, Ω is the quadrilateral $QRUW$, which is a trapping region:



We first prove that the geometry of $T^2(\mathfrak{Q})$ is the same as the picture:



This allows us to prove Lemmas 5.16, 5.17, 5.18, 5.19, 5.20, and 5.21.

We do this by finding the vertical and horizontal tangents of $T^2(\overline{QW})$, $T^2(\overline{WU})$, $T^2(\overline{RU})$, $T^2(\overline{QR})$, and knowing the approximate location of the points $T^2(Q)$,$T^2(R)$, $T^2(W)$, and $T^2(U)$. A simple computation shows that $T^2(Q) \approx (-.962, -.317)$, $T^2(R) \approx (-.930, -.322)$, $T^2(W) \approx (-.993, -.392)$, and $T^2(U) \approx (-1.029, -.393)$, where the exact x and y coordinates of these points are within $\epsilon = .001$ of the values written above. The second iterate of the Henon map is

$$T^2(x,y) = T(1+y-ax^2, bx) = (1+bx-a[1+y-ax^2]^2, b[1+y-ax^2]).$$

We now find where the curve $T^2(\overline{QW})$ has a vertical tangent. We parameterize the line segment between points $Q = (q_1, q_2)$ and $W = (w_1, w_2)$:
$tQ+(1-t)W = (t(q_1-w_1)+w_1, t(q_2-w_2)+w_2)$. We substitute this parameterization into $T^2(x,y)$ and find the vertical tangents i.e. when $\frac{dx}{dy} = \frac{dx}{dt}\frac{dt}{dy} = 0$. This condition occurs when $\frac{dx}{dt} = 0$ and $t \in [0,1]$. The x coordinate of $T^2((tq_1 + (1-t)w_1, tq_2 + (1-t)w_2))$ is:
$x(t) = 1 + b[t(q_1-w_1)+w_1] - a[1+w_2+(q_2-w_2)t - a[t(q_1-w_1)+w_1]^2]^2$.

The derivative is $\frac{dx}{dt} =$
$b(q_1-w_1) - 2a[1+w_2+(q_2-w_2)t - a[t(q_1-w_1)+w_1]^2]$
$[(q_2-w_2) - 2a(t(q_1-w_1)+w_1)(q_1-w_1)]$.

In order to find the vertical tangents, we set $\frac{dx}{dt} = 0$, so we see that

$b(q_1-w_1) =$
$2a[1+w_2+(q_2-w_2)t - a[t^2(q_1-w_1)^2 + 2t(q_1-w_1)w_1 + w_1{}^2]$
$[(q_2-w_2) - 2aw_1(q_1-w_1) - 2a(q_1-w_1)^2t]]$.

Thus, $b(q_1-w_1) =$
$2a[(1+w_2) - aw_1{}^2 + [(q_2-w_2) - 2aw_1(q_1-w_1)]t - a(q_1-w_1)^2t^2]$
$[(q_2-w_2) - 2aw_1(q_1-w_1) - 2a(q_1-w_1)^2t]]$.

Hence, $b(q_1-w_1) =$
$[2a(1+w_2) - 2a^2w_1{}^2 + [2a(q_2-w_2) - 4a^2w_1(q_1-w_1)]t - 2a^2(q_1-w_1)^2t^2]$
$[(q_2-w_2) - 2aw_1(q_1-w_1) - 2a(q_1-w_1)^2t]]$.

Set $L = 2a^2(q_1 - w_1)^2$.

Set $M = 4a^2w_1(q_1 - w_1) - 2a(q_2 - w_2)$.

Set $N = 2a^2{w_1}^2 - 2a(1 + w_2)$.

Set $E = -2a(q_1 - w_1)^2$.

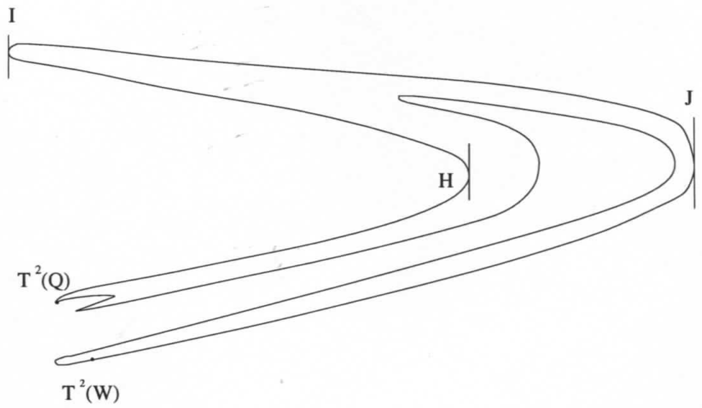
Set $F = (q_2 - w_2) - 2aw_1(q_1 - w_1)$.

We are left with a cubic equation,

$0 = \frac{dx}{dt} = LEt^3 + (LF + ME)t^2 + (MF + EN)t + FN + b(q_1 - w_1)$. Notice that $\frac{dx}{dt}(.1), \frac{dx}{dt}(.6), \frac{dx}{dt}(.8) > 0$ and that $\frac{dx}{dt}(.4), \frac{dx}{dt}(.5), \frac{dx}{dt}(.9) < 0$ so the Intermediate Value Theorem implies that the curve $T^2(\overline{QW})$ has three solutions of $\frac{dx}{dt} = 0$ between the appropriate values of t.

A simple computation shows these vertical tangents are located at the points: $J = (1.28, -.01)$, $I = (-1.30, .38)$, and $H = (.70, .01)$, where the x and y values above are within $\epsilon = .01$ of the actual x, and y coordinates of the vertical tangents.

Vertical Tangents of the image of segment QW

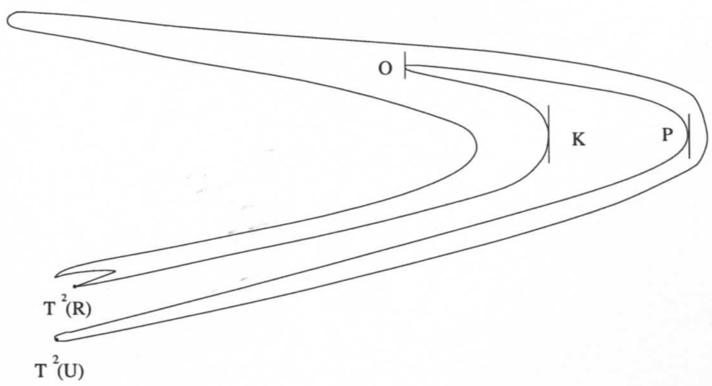


We observe that $y(t) = b[1 + tq_2 + (1-t)w_2 - a(tq_1 + (1-t)w_1)^2]$ and $\frac{dy}{dt} = b[q_2 - w_2 - 2a(tq_1 + (1-t)w_1)(q_1 - w_1)]$.

If we set $\frac{dy}{dt} = 0$ and solve for the critical point t_0, we obtain $t_0 = \frac{(q_2-w_2)}{2a(q_1-w_1)^2} - \frac{w_1}{(q_1-w_1)}$ and $\frac{d^2y}{dt^2} = -2ab[q_1 - w_1]^2 < 0$. Hence, $\frac{dy}{dt} > 0$ when $t \in [0, t_0)$ and $\frac{dy}{dt} < 0$ when $t \in (t_0, 1]$. The point where the horizontal tangent to the curve $T^2(\overline{QW})$ occurs is $(-1.298, .383)$ where the x and y coordinates of this point are within $\epsilon = .001$ of the actual x, and y coordinates of the horizontal tangent point.

We apply the same methods to the curve $T^2(\overline{RU})$. We replace the point Q by the point R, and replacing the point W by the point U in the parameterized equations. After a similar analysis, one finds three vertical tangents to the curve $T^2(\overline{RU})$, at the points $P = (1.228, -.016)$, $O = (.380, .200)$, and $K = (.814, .017)$, where the x and y values above are within $\epsilon = .001$ of the actual x, and y coordinates of the vertical tangent point.

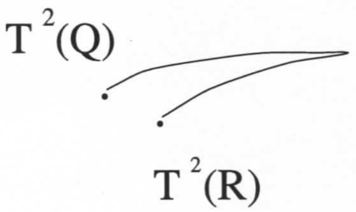
Vertical Tangents of the image of segment RU



The horizontal tangent to the curve $T^2(\overline{RU})$, occurs at the point $(.388, .201)$, where the x and y coordinates of these points are within $\epsilon = .001$ of the actual x, and y coordinates of the horizontal tangent point.

Next we analyze $T^2(\overline{RQ})$. We can parameterize the curve $T^2(\overline{RQ})$, as before, and we see that the x coordinate of this curve is

$x(t) = 1 + b[t(q_1 - r_1) + r_1] - a[1 + r_2 + (q_2 - r_2)t - a[t(q_1 - r_1) + r_1]^2]^2$. There is one vertical tangent at the point, $(-.870, -.312)$ where $\epsilon = .001$. We note that this tangent occurs when $t = t_0$, and $t_0 \approx .555$. Thus, $\frac{dx}{dt} > 0$ when $t \in [0, t_0)$, and $\frac{dx}{dt} < 0$ when $t \in (t_0, 1]$. We have $x(0) = T^2(R)$, and $x(1) = T^2(Q)$. Hence, we have verified that $T^2(\overline{QR})$ is a curve that has the following geometry:



Now we analyze $T^2(\overline{WU})$. We can parameterize the curve $T^2(\overline{WU})$, as before. The x coordinate of this curve is

$x(t) = 1 + b[t(u_1 - w_1) + w_1] - a[1 + w_2 + (u_2 - w_2)t - a[t(u_1 - w_1) + w_1]^2]^2$. Using the same analysis as before, we see that $\frac{dx}{dt} > 0$ for all $t \in [0, 1]$. We see that the y coordinate of the curve $T^2(\overline{WU})$ is $y(t) = b[1 + tu_2 + (1-t)w_2 - a(tu_1 + (1-t)w_1)^2]$, and the derivative is $\frac{dy}{dt} = b[u_2 - w_2 - 2a(tu_1 + (1-t)w_1)(u_1 - w_1)]$. Thus, $\frac{dy}{dt} = 0$ at exactly one point which is $(-.997, -.392)$, where $\epsilon = .001$. Further, $\frac{d^2y}{dt^2} = -2ab[u_1 - w_1]^2 < 0$.

REMARK 5.29. *The picture illustrating the geometry and topology of $T^2(QRUW)$ is justified.*

Proof: Since T^2 is a diffeomorphism, $T^2(\overline{QR})$, and $T^2(\overline{WU})$, do not intersect. We know the location of the vertical tangents, what the sign of $\frac{dx}{dt}$, and $\frac{dy}{dt}$ are along $T^2(\overline{QW})$, $T^2(\overline{WU})$, $T^2(\overline{QR})$, and $T^2(\overline{RU})$, and the coordinates of the points $T^2(Q), T^2(W), T^2(R), T^2(U)$. Combining all of this information verifies the picture.

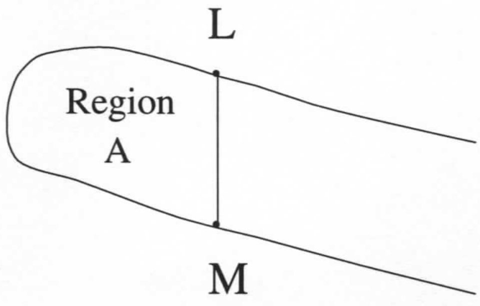
■

LEMMA 5.26. *$T[(x = -1.1) \cap$ Region $A] \subset$ Region C.*

Proof: We solve the parametric equation $T^2(\overline{QW})$ when $t \in [0, 1]$. This equation $x(t) = 1 + b[t(w_1 - q_1) + q_1] - a[1 + q_2 + (w_2 - q_2)t - a[t(w_1 - q_1) + q_1]^2]^2$ has two solutions, t_1 and t_2. Since we can not solve exactly for t_1 and t_2, we find t values near t_1, and t_2, and then use the intermediate value theorem. In particular, we have $x(.395) > -1.09$. $x(.40) < -1.10$, so $x(.40) < -1.1 < x(.395)$. The intermediate value theorem implies there is a solution t_1 such that $x(t_1) = -1.1$, and $0.395 < t_1 < 0.40$. Since $\frac{dy}{dt} > 0$ on this part of the curve $T^2(\overline{QW})$, y is increasing as a function of t, we know that $y(0.40) > y(t_1)$. Hence, $y(t_1) < 0.37$.

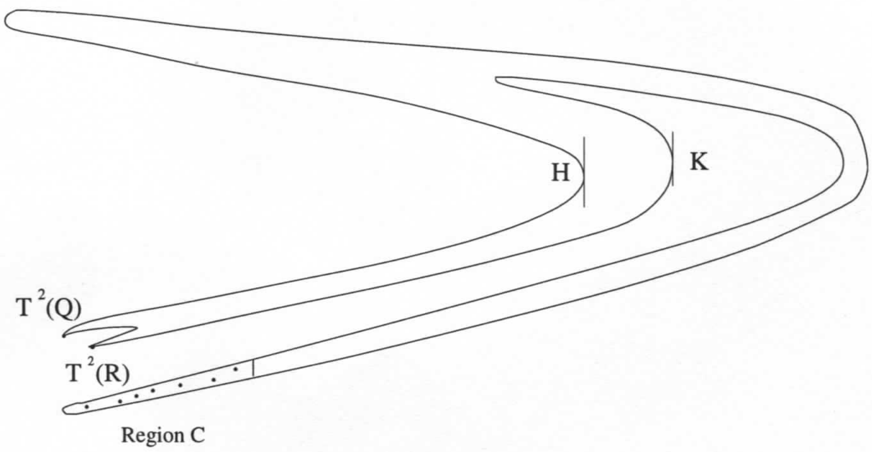
Now we solve for an estimate of t_2. We have $x(.55) < -1.1$ and $x(.555) > -1.1$, so $x(.55) < -1.1 < x(.555)$. By the intermediate value theorem, we have a solution t_2 such that $x(t_2) = -1.1$, and $0.55 < t_2 < 0.555$. Since $\frac{dy}{dt} < 0$ on this part of the curve $T^2(\overline{QW})$, we know that y is decreasing as a function of t. Hence, we obtain the estimate $y(t_2) < y(.555) < .375$.

Define $y_1 = y(t_1)$ and $y_2 = y(t_2)$. For simplicity, define the points $L = (-1.1, y_1)$ and $M = (-1.1, y_2)$. The set $(x = -1.1) \cap$ Region A is the vertical line segment $\overline{LM}$.



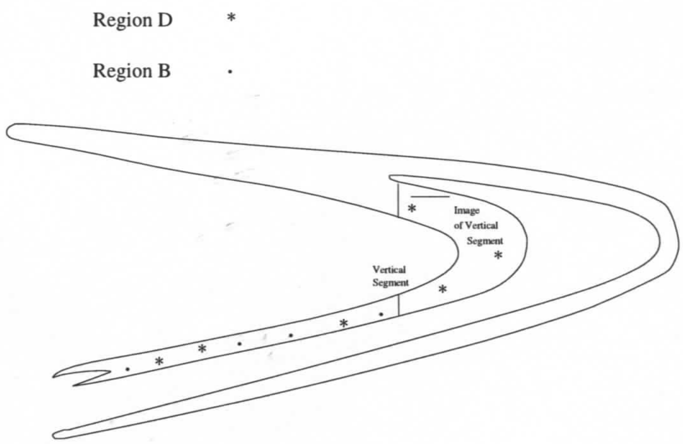
Since T maps a vertical line to a horizontal line, if we refer to the definition of Region C, we must show that $1 + y - a(-1.1)^2 \leq -.31$ for every y value occurring in $(x = -1.1) \cap$ Region A. However, from above, $y_1, y_2 < .375$ implies that $1 + y - a(-1.1)^2 < 1 + .375 - (1.4)(-1.1)^2 < -.31$.

Now we verify that the image of $(x = -1.1) \cap$ Region A lies in the lowest section of $T^2(\mathfrak{Q})$. The y coordinate of any image point is $b * (-1.1) = -.33$. Thus, the horizontal line $T(\overline{LM})$ has a y value equal to $-.33$. Observe that $T^2(Q) \approx (-.96, -.317)$, and $T^2(R) \approx (-.93, -.322)$. The slope of the curve from $T^2(Q)$ to the vertical tangent at point H is increasing; the slope of the curve from $T^2(R)$ to the vertical tangent at point K is increasing.



Since $T(\overline{LM}) \subset T^2(\mathfrak{Q})$, this implies that $T(\overline{LM})$ must lie in Region C. ■

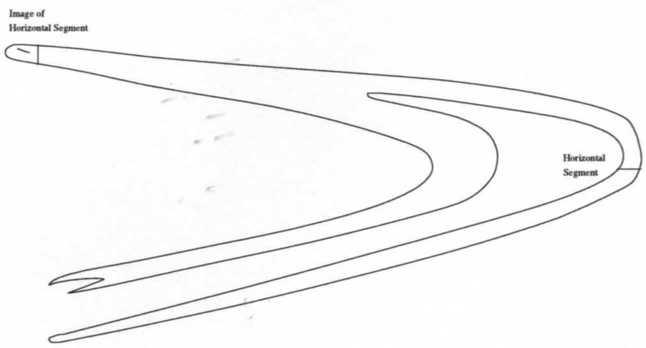
LEMMA 5.27. *$T[(x = .56) \cap$ Region $B] \subset$ Region D.*



Proof: We solve the parametric equation $T^2(\overline{QW})$ when $t \in [0,1]$, and we solve $T^2(\overline{RU})$ when $t \in [0,1]$. Thus, we bound the vertical line segment $(x = .56)$ $\cap$ Region B between two y values. When $t = .085$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{QW})$ satisfies the inequalities $.53 < x < .54$, and $-.09 > y > -.10$. When $t = .09$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{QW})$ satisfies the inequalities $.56 < x < .57$, and $-.08 > y > -.09$. When $t = .105$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{RU})$ satisfies the inequalities $.53 < x < .54$, and $-.12 < y < -.11$. When $t = .11$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{RU})$ satisfies the inequalities $.56 < x < .57$, and $-.12 < y < -.11$. These inequalities imply that the vertical line segment $(x = .56) \cap$ Region B is a subset of the vertical line segment $\{.56\} \times [-.08, -.12]$.

A simple computation shows that $T(\{.56\} \times [-.08, -.12] \cap$ Region $B)$ is a subset of the horizontal line segment $[.43, .50] \times \{.168\} \cap \mathfrak{A}$. This horizontal line segment lies inside of Region D. ■

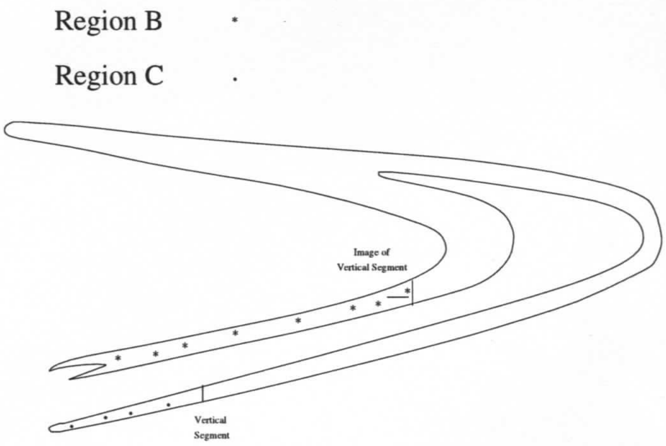
LEMMA 5.28. *$T([1.22, 1.28] \times \{-0.02\} \cap$ Region $E) \subset$ Region A.*



Proof: From what we know about the location of the vertical tangents, and the points $T^2(Q), T^2(U), T^2(W)$, and $T^2(R)$, the only part of $T^2(\Omega)$ that lies to the left of the vertical line $x = -1.1$ is Region A. Hence, we need to show that the image of this horizontal line segment lies to the left of the vertical line $x = -1.1$.

A simple computation shows that the point $T(1.22, -0.02)$ has an x coordinate less than -1.1. Hence, all points on the horizontal line segment $[1.22, 1.28] \times \{-0.02\} \cap$ Region E must lie to the left of $x = -1.1$. ■

LEMMA 5.29. *$T[(x = -.31) \cap$ Region $C] \subset$ Region B.*

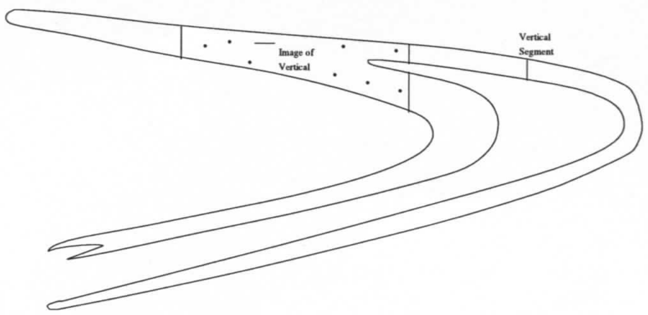


Proof: When $t = .97$ the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{RU})$ satisfies the inequalities $-.29 > x > -.30$, and $-.32 > y > -.33$. When $t = .975$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{RU})$ satisfies the inequalities $-.40 > x > -.41$, and $-.33 > x > -.34$. When $t = .975$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{QW})$ satisfies the inequalities $-.19 > x > -.20$, and $-.31 > y > -.32$. When $t = .98$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{QW})$ satisfies the inequalities $-.34 > x > -.35$, and $-.33 > y > -.34$. These inequalities imply that the vertical line segment $(x = -.31) \cap$ Region $C]$ is a subset of the vertical line segment $\{-.31\} \times [-.31, -.34]$.

The image of this vertical line segment is a horizontal line segment that is a subset of the horizontal line segment: $[.52, .56] \times \{-.093\}$. By the definition of Region B, the horizontal line segment $\{[.52, .56] \times \{-.093\}\} \cap T^2(\Omega)$ is a subset of Region B. ■

LEMMA 5.30. *$T((x = 0.9) \cap$ Region $E) \subset$ Region F.*

Region F



Proof: When $t = .665$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{RU})$ satisfies the inequalities $.88 < x < .89$, and $.12 < y < .13$. When $t = .67$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{RU})$ satisfies the inequalities $.90 < x < .91$, and $.12 < y < .13$. When $t = .77$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{QW})$ satisfies the inequalities $.87 < x < .88$, and $.14 < y < .15$. When $t = .775$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{QW})$ satisfies the inequalities $.91 < x < .92$, and $.13 < y < .14$. These inequalities imply that the vertical line segment $(x = .9) \cap$ Region E is a subset of the vertical line segment $\{.9\} \times [.12, .15]$.

The image of this vertical line segment is a horizontal line segment that is a subset of the horizontal line segment $[-.05, .05] \times \{0.27\}$. By the definition of Region F, $[-.05, .05] \times \{0.27\}$ is a subset of Region F. ■

LEMMA 5.31. *$T((x = 0.40) \cap$ Region $D) \subset$ Region E.*

Proof: When $t = .420$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{RU})$ satisfies the inequalities $.42 < x < .43$, and $.14 < y < .15$. When $t = .421$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{RU})$ satisfies the inequalities $.39 < x < .40$, and $.19 < y < .20$. When $t = .205$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{QW})$ satisfies the inequalities $.42 < x < .43$, and $.14 < y < .15$. When $t = .21$, the point $(x(t), y(t))$ on the parametric curve $T^2(\overline{QW})$ satisfies the inequalities $.38 < x < .39$, and $.15 < y < .16$. These inequalities imply that the vertical line segment $(x = .4) \cap$ Region D is a subset of the vertical line segment $\{.4\} \times [.14, .20]$.

The image of the vertical line segment $\{.4\} \times [.14, .20]$ is a horizontal line segment that is a subset of the horizontal line segment $[.91, .98] \times \{0.12\}$. By the definition of Region E, the vertical line segment $[.91, .98] \times \{0.12\}$ is a subset of Region E. ■