THEOREM 4.17. *Suppose (X, d) is a compact metric space. Suppose $f_1, f_2, \dots, f_n : X \longrightarrow X$ are continuous functions. Then $h(f_n \circ f_{n-1} \circ \dots f_2 \circ f_1) = h(f_k \circ f_{k-1} \dots f_1 \circ f_n \circ f_{n-1} \dots f_{k+1})$ for any k satisfying $1 \leq k < n$.*

Proof: Set $g = f_n \circ f_{n-1} \dots f_2$, and set $f = f_1$. Because of Corollary 4.5, we have $h(f_n \circ f_{n-1} \circ \dots f_2 \circ f_1) = h(f_1 \circ f_n \circ f_{n-1} \dots f_2)$. Now set $f = f_2$ and $g = f_1 \circ f_n \circ f_{n-1} \dots f_3)$ and apply Corollary 4.5; this implies $h(f_1 \circ f_n \circ f_{n-1} \dots f_2) = h(f_2 \circ f_1 \circ f_n \circ f_{n-1} \dots f_3)$. By induction, we continue making this argument until we cycle all the way around. ■

Next we develop a few lemmas so that we can generalize $h(\{f, g\}) = \frac{1}{2}h(g \circ f)$ to $h(\{f_1, f_2, \dots, f_p\}) = \frac{1}{p}h(f_p \circ f_{p-1} \dots f_2 \circ f_1)$.

LEMMA 4.12. *Suppose $(X, \{f_1, f_2, \dots, f_p\})$ is a non-autonomous system with period p. For any $\epsilon > 0$ there exists $\delta > 0$, independent of n, satisfying the statment: If T is a $(0, n, \delta, f_p \circ f_{p-1} \dots f_2 \circ f_1)$ spanning set for X, then T is a $(0, pn, \epsilon, \{f_1, f_2, \dots, f_p\})$ spanning set for X.*

Proof: Let $\epsilon > 0$. Then since f_p is uniformly continuous there exists $\delta_p > 0$ such that $d(z, w) < \delta_p$ implies $d(f_p(z), f_p(w)) < \epsilon$. Since f_{p-1} is uniformly continuous, there exists $\delta_{p-1} > 0$ such that for any $z, w \in X$, $d(z, w) < \delta_{p-1}$ implies $d(f_{p-1}(z), f_{p-1}(w)) < \delta_p$. Inductively, we can construct $\delta_{k-1} > 0$ such that $d(z, w) < \delta_{k-1}$ which implies that $d(f_{k-1}(z), f_{k-1}(w)) < \delta_k$. Set $\delta = \min\{\delta_1, \delta_2, \delta_3, \dots, \delta_p, \epsilon\}$. Then $\delta > 0$. Suppose the set T is a $(0, n, \delta, f_p \circ f_{p-1}, \dots, f_2 \circ f_1)$ spanning set for X Let $z \in X$. By the definition of T there exist $s \in T$ so that

$$(4.9) \qquad d([f_p \circ f_{p-1} \circ \ldots f_2 \circ f_1]^k(s), [f_p \circ f_{p-1} \circ \ldots f_2 \circ f_1]^k(z)) < \delta.$$

Let j be any integer satisfying $0 \leq j < pn$. Then $j = pk + r$ for some $0 \leq r < p$. Recall that $[f_p, f_{p-1}, \ldots, f_1]^{pk+r} = f_r \circ f_{r-1} \circ \ldots f_2 \circ f_1 \circ (f_p \circ \cdots \circ f_2 \circ f_1)^k$. Hence, $d([f_p, f_{p-1}, \ldots, f_1]^j(s), [f_p, f_{p-1}, \ldots, f_1]^j(x)) =$
$d(f_r \circ \ldots f_2 \circ f_1 \circ (f_p \circ f_{p-1} \cdots \circ f_1)^k(s), f_r \circ \ldots f_2 \circ f_1 \circ (f_p \circ f_{p-1} \cdots \circ f_1)^k(x)) < \epsilon$, by 4.9 and the definition of δ. ■

REMARK 4.17. *For any $\epsilon > 0$ there exists $\delta > 0$ independent of n so that $r_{span}(0, pn, \epsilon, \{f_1, f_2, \ldots, f_p\}) \leq r_{span}(0, n, \delta, f_p \circ f_{p-1} \ldots f_2 \circ f_1)$.*

Proof: This follows immediately from Lemma 4.12 and the fact that $r_{span} =$ the number of elements in a minimal spanning set.

LEMMA 4.13. *For any $\epsilon > 0$ there exists $\delta > 0$ so that $h(\epsilon, \{f_1, f_2, \ldots, f_p\}) \leq \frac{1}{p} h(\delta, f_p \circ f_{p-1} \ldots f_2 \circ f_1)$.*

Proof: Let $\epsilon > 0$. From Remark 4.17

$$\frac{\log r_{span}(0, pn, \epsilon, \{f_1, f_2, \ldots, f_p\})}{pn} \leq \frac{\log r_{span}(0, n, \delta, f_p \circ f_{p-1} \ldots f_2 \circ f_1)}{pn}$$

The inequality implies that

$$h(\epsilon, \{f_1, f_2, \ldots, f_p\}) \leq \limsup_{n \to \infty} \frac{\log r_{span}(0, pn, \epsilon, \{f_1, f_2, \ldots, f_p\})}{pn}.$$

The right hand expression is less than or equal to:

$$\frac{1}{p}\limsup_{n\to\infty}\frac{\log r_{span}(0,n,\epsilon,\{f_p\circ f_{p-1}\cdots\circ f_1\})}{n}=\frac{1}{p}h(\delta,\{f_p\circ f_{p-1}\cdots\circ f_1\}).$$

■

LEMMA 4.14. *The first inequality is* $h(\{f_1,f_2,\ldots,f_p\})\leq\frac{1}{p}h(\delta,\{f_p\circ f_{p-1}\cdots\circ f_1\})$.

Proof: For any ϵ_1,ϵ_2 satisfying $0<\epsilon_1<\epsilon_2$, $h(\epsilon_1,\{g_i\})\geq h(\epsilon_2,\{g_i\})$. Hence, Lemma 4.13 yields the result. ■

Now we work toward the inequality in the opposite direction; i.e. $h(\{f_1,\ldots,f_p\})\geq\frac{1}{p}h(f_p\circ f_{p-1}\ldots f_2\circ f_1)$.

REMARK 4.18. *Suppose* T *is* $(0,n,\epsilon,f_p\circ\ldots f_2\circ f_1)$ *separated, then* T *is a* $(0,pn,\epsilon,\{f_1,f_2,\ldots f_p\})$ *separated set.*

Proof: Let $x,y\in T$. By definition, there is a k so that $d((f_p\circ\ldots f_2\circ f_1)^k(x),(f_p\circ\ldots f_2\circ f_1)^k(y))>\epsilon$, then $d([f_p,\ldots f_2\circ f_1]^{pk}(x),[f_p,\ldots f_2\circ f_1]^{pk}(y))>\epsilon$ and $0\leq pk<pn$. ■

REMARK 4.19. *An intermediate inequality is* $r_{sep}(0,pn,\epsilon,\{f_1,f_2,\ldots f_p\})\geq r_{sep}(0,n,\epsilon,f_p\circ\ldots f_2\circ f_1)$.

Proof: This is immediate from Remark 4.18. ■

LEMMA 4.15. *For any* $\epsilon > 0$, $h(\epsilon, \{f_1, \ldots, f_p\}) \geq \frac{1}{p} h(\epsilon, f_p \circ f_{p-1} \circ \ldots f_2 \circ f_1)$.

Proof: By Remark 4.19, for all n,

$$\frac{\log r_{sep}(0, pn, \epsilon, \{f_1, f_2, \ldots, f_p\})}{pn} \geq \frac{\log r_{sep}(0, n, \epsilon, f_p \circ f_{p-1} \ldots f_2 \circ f_1)}{pn}.$$

Take the $\limsup_{m \to \infty}$ of both sides of this inequality:

$$\begin{aligned} h(\epsilon, \{f_1, f_2, \ldots, f_p\}) &= \limsup_{m \to \infty} \tfrac{\log r_{sep}(0, m, \epsilon, \{f_1, f_2, \ldots, f_p\})}{m} \\ &\geq \limsup_{n \to \infty} \tfrac{\log r_{sep}(0, pn, \epsilon, \{f_1, f_2, \ldots, f_p\})}{pn} \\ &\geq \tfrac{1}{p} \limsup_{n \to \infty} \tfrac{\log r_{sep}(0, n, \epsilon, f_p \circ f_{p-1} \ldots f_2 \circ f_1)}{n} \\ &\geq \tfrac{1}{p} h(\epsilon, f_p \circ f_{p-1} \ldots f_2 \circ f_1). \ \blacksquare \end{aligned}$$

THEOREM 4.18. *If* $f_1, f_2, \ldots, f_p : X \longrightarrow X$ *are continuous functions, then the following relationship holds between the entropy of the non-autonomous system* $(X, \{f_1, f_2, \ldots, f_p\})$ *with period* p, *and the entropy of the autonomous system* $(X, f_p \circ f_{p-1} \circ \ldots f_2 \circ f_1)$

$$h(\{f_1, \ldots, f_p\}) = \frac{1}{p} h(f_p \circ f_{p-1} \ldots f_2 \circ f_1).$$

Proof: From Lemma 4.15, $h(\epsilon, \{f_1, \ldots, f_p\}) \geq \frac{1}{p} h(f_p \circ f_{p-1} \ldots f_2 \circ f_1)$. Take the $\lim_{\epsilon \to 0}$ of both sides to obtain $h(\{f_1, \ldots, f_p\}) \geq \frac{1}{p} h(f_p \circ f_{p-1} \ldots f_2 \circ f_1)$. The result is immediate by applying Lemma 4.14. $\blacksquare$

The next idea involves reducing topological entropy of a non-autonomous system to computing the topological entropy of an autonomous system. Consider the period 2 non-autonomous system $(X, \{f, g, f, g, \ldots\}$. The idea is to find the square root of $g \circ f$ with respect to function composition i.e. find a continuous function S so that $S \circ S = g \circ f$.

THEOREM 4.19. *Suppose $f, g, S : X \longrightarrow X$ where X is a compact metric space. f, g, S are continuous, and $S \circ S = g \circ f$. Then the topological entropy of S equals the topological entropy of $\{f, g, f, g, \ldots\}$. In our notation $h(S) = h(\{f, g\})$.*

Proof: The idea here is to use the uniform continuity of f, S, (X is compact), and then utilize the spanning set definition of topological entropy.

Let $\gamma > 0$. Since f is uniformly continuous, there exists $\delta_f(\gamma) > 0$ satisfying $\delta_f(\gamma) \leq \gamma$ and $d(x, y) < \delta_f(\gamma)$ implies that $d(f(x), f(y)) < \gamma$ for any $x, y \in X$. Similarly, since S is uniformly continuous, there exists $\delta_S(\gamma) > 0$ and $\delta_S(\gamma) \leq \gamma$ so that $d(x, y) < \delta_S(\gamma)$ implies that $d(S(x), S(y)) < \gamma$ for any $x, y \in X$.

Now we develop four Remarks so that we can finish the proof.

REMARK 4.20. *If the set T $(j, n, \delta_f(\gamma), S)$ spans X, then T also $(j, n, \gamma, \{f, g\})$ spans X, whenever $\delta_f(\gamma) \leq \gamma$.*

Proof: Let $y \in X$. Then there is an $x \in T$ so that $d(x, y) < \delta_f(\gamma)$ and $d(S^i(x), S^i(y)) < \delta_f(\gamma)$ for $j \leq i < n$. When i is even, $i = 2k$ and $0 \leq i < n$, then $d([g, f]^i(x), [g, f]^i(y)) = d(S^i(x), S^i(y)) < \delta_f(\gamma) \leq \gamma$ When i is odd, $i = 2k + 1$ and $0 \leq i < n$, then $d([g, f]^i(x), [g, f]^i(y)) = d(f \circ S^k(x), f \circ S^k(y)) < \gamma$ because $d(S^k(x), S^k(y)) < \delta_f(\gamma)$. ■

REMARK 4.21. *If the set T $(j, n, \delta_S(\gamma), \{f, g\})$ spans X, then T (j, n, γ, S) spans X, whenever $\delta_S(\gamma) \leq \gamma$.*

Proof: Let $y \in X$. Then there is an $x \in T$ so that $d([g,f]^i(x),[g,f]^i(y)) < \delta_S(\gamma)$ for $j \leq i < n$. When k is even, $k = 2m$ and $j \leq k < n$, then $d(S^k(x), S^k(y)) = d((g \circ f)^m(x), (g \circ f)^m(y)) < \delta_S(\gamma) \leq \gamma$. When k is odd, $k = 2m+1$ and $j \leq k < n$, then $d(S^k(x), S^k(y)) = d(S \circ [g,f]^{2m}(x), S \circ [g,f]^{2m}(y)) < \gamma$ by the definition of $\delta_S(\gamma)$. ■

REMARK 4.22. *For any* $\gamma > 0$, $r_{span}(j,n,\gamma,\{f,g\}) \leq r_{span}(j,n,\delta_f(\gamma),S)$ *whenever* $\delta_f(\gamma) \leq \gamma$.

Proof: This Remark immediately follows from Remark 4.20. ■

REMARK 4.23. *For any* $\gamma > 0$, $r_{span}(j,n,\gamma,S) \leq r_{span}(j,n,\delta_S(\gamma),\{f,g\})$ *whenever* $\delta_S(\gamma) \leq \gamma$.

Proof: This Remark immediately follows from Remark 4.20. ■

Now that we have established the four Remarks, we finish the proof. Fix $\epsilon > 0$. Then by the above we were able to choose $\delta_f(\epsilon) \leq \epsilon$. Hence, $h(\epsilon,\{f,g\}) \leq h(\delta_f(\epsilon), S)$ by Remark 4.22. Since $\epsilon > 0$ was arbitrary and we were able to find an $\eta = \delta_f(\epsilon)$ so that $h(\epsilon,\{f,g\}) \leq h(\eta, S)$ we see that $h(\{f,g\}) \leq h(S)$.

Similarly, fix $\epsilon > 0$. By the previous we were able to choose $\delta_S(\epsilon) \leq \epsilon$. Hence, $h(\epsilon, S) \leq h(\delta_f(\epsilon),\{f,g\})$ by Remark 4.23. Since $\epsilon > 0$ was arbitrary and we were able to find an $\eta = \delta_S(\epsilon)$ so that $h(\epsilon, S) \leq h(\eta,\{f,g\})$ we see that $h(S) \leq h(\{f,g\})$.

■

This idea can be extended to the nth root i.e. the existence of an S so that $S^n = g_n \circ \dots g_2 \circ g_1$. We now show the uniqueness of square roots of period 2 functions modulo topological entropy.

THEOREM 4.20. *Suppose $f, g : X \longrightarrow X$ where (X, d) is a compact metric space. Suppose there exists continuous functions $S, R : X \longrightarrow X$ where $S \circ S = g \circ f$ and $R \circ R = f \circ g$. Then we have $h(S) = h(R)$.*

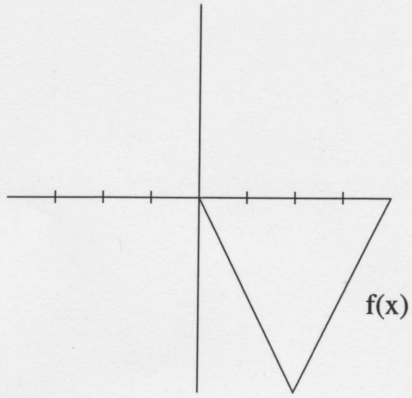
Proof: By theorem 4.19, $h(S) = h(\{f, g\})$ and $h(R) = h(\{g, f\})$. By theorem 4.15, $h(\{f, g\}) = h(\{g, f\})$. Hence, $h(S) = h(R)$. ■

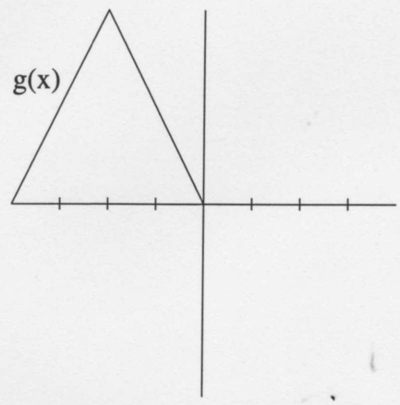
This next section explores a few examples that offer insight on the theorems just proven. A natural question is what is the relationship between $h(\{f, g, \dots\})$ and $h(f)$ and $h(g)$. The following example shows that we can choose f and g so that $h(f) = h(g) = 0$, but $h(\{f, g, \dots\}) > 0$. In fact, using the technique in this example we can make $h(\{f, g, \dots\})$ arbitrarily large, yet $h(f) = h(g) = 0$.

Set $X = [-1, 1]$ and define $f, g : X \to X$ where

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0, \\ -2x & \text{if } 0 < x \le \frac{1}{2}, \\ 2x - 2 & \text{if } \frac{1}{2} < x \le 1, \end{cases}$$

$$g(x) = \begin{cases} 2x + 2 & \text{if } -1 \le x \le -\frac{1}{2}, \\ -2x & \text{if } -\frac{1}{2} \le x \le 0, \\ 0 & \text{if } 0 \le x \le 1, \end{cases}$$





$$g \circ f(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0, \\ 4x & \text{if } 0 \leq x \leq \frac{1}{4}, \\ 2 - 4x & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 4x - 2 & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4}, \\ -4x + 4 & \text{if } \frac{3}{4} \leq x \leq 1. \end{cases}$$

First, we note that $f([0,1]) = [-1,0]$. Hence, $f^2([0,1]) = \{0\}$, and $f([-1,0]) = \{0\}$. Hence, $k \geq 2$ implies that f^k is the constant 0 function, so $h(f) = 0$. We make a similar argument for g. Hence, $g([-1,0]) = [0,1]$, so $g^k = 0$ whenever $k \geq 2$. Consequently, $h(g) = 0$.

Thus, $(g \circ f)|_{[0,1]} = T \circ T$ where T is the tent map. Hence, $h(T^2) = 2h(T) = 2log2$, so $h(\{f, g, \dots\}) \geq \frac{1}{2}h(g \circ f) \geq \frac{1}{2}h(T^2) = \log 2$. Hence, we see that $h(\{f, g\}) >$

$\max\{h(f), h(g)\} = 0$. In fact, by making 2^n tents in the same region that we made tents in the definition of f and g, we see that we can make $h(\{f, g\})$ arbitrarily large, yet $h(f) = h(g) = 0$.

Suppose r, l, α are continuous functions. The following example illustrates that $h(l \circ \alpha) = h(\alpha \circ l)$ does not imply that $h(r \circ l \circ \alpha) = h(r \circ \alpha \circ l)$. In other words, $h(f) = h(g)$ does not imply that $h(r \circ f) = h(r \circ g)$. Define $\alpha, l, r : [0,1] \longrightarrow [0,1]$ where

$$\alpha(x) = \begin{cases} \frac{1}{2} & if \quad 0 \le x \le \frac{1}{2}, \\ x & if \quad \frac{1}{2} < x \le 1, \end{cases}$$

$$l(x) = \begin{cases} 4x & if \quad 0 \le x \le \frac{1}{4}, \\ 2 - 4x & if \quad \frac{1}{4} < x \le \frac{1}{2}, \\ 0 & if \quad \frac{1}{2} < x \le 1, \end{cases}$$

$$r(x) = \begin{cases} 0 & if \quad 0 \le x \le \frac{1}{2}, \\ 4x - 2 & if \quad \frac{1}{2} < x \le \frac{3}{4}, \\ 4 - 4x & if \quad \frac{3}{4} < x \le 1. \end{cases}$$

Note that $h(r \circ l \circ \alpha) = 0$, but $h(r \circ \alpha \circ l) = \log 4$. Set $\sigma_1 = [0, \frac{1}{4}]$, $\sigma_2 = [\frac{1}{4}, \frac{1}{2}]$, $\sigma_3 = [\frac{1}{2}, \frac{3}{4}]$, $\sigma_4 = [\frac{3}{4}, 1]$. We can define a simplicial matrix for each map α l and r. Let $M(\alpha) = (m_{ij})$ be a four by four matrix, where $m_{ij} = 1$ if $\sigma_j \subset \alpha(\sigma_i)$, and $m_{ij} = 0$, otherwise. Define $M(l)$ and $M(r)$ in a similar way.

Now consider the simplicial matrices for α, l, and r.

$$M(\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$h(g \circ f \circ f \circ g) = h(g \circ g \circ f \circ f) = h(f \circ f \circ g \circ g)$ by Lemma 4.5. However, the word $ffgg$ is not the same as $fgfg$ as the following example illustrates:

Define $f, g : [0,1] \longrightarrow [0,1]$ where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 4x - 2 & \text{if } \frac{1}{2} < x \leq \frac{3}{4}, \\ 4 - 4x & \text{if } \frac{3}{4} < x \leq 1, \end{cases}$$

$$g(x) = \begin{cases} 2x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{4}, \\ \frac{3}{2} - 2x & \text{if } \frac{1}{4} < x \leq \frac{1}{2}, \\ \frac{1}{2} & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Then $g \circ g$ is a constant map, so $h(f \circ f \circ g \circ g) = 0$. However, $h(f \circ g \circ f \circ g) = 2h(f \circ g) = 2 \log 4 = 4 \log 2$ because $f \circ g$ has a graph with two tents.