Dynamical Equivalence: comparing different non-autonomous systems

In the Introduction, it is explained why the training process in neural networks can be modeled as a non-autonomous iterative dynamical system. (We define non-autonomous in the first definition.) In this section, we introduce a way to compare two different non-autonomous dynamical systems, and we recall and define concepts such as non-wandering and periodic. The first introduced concept is dynamical equivalence; dynamical equivalence is a way to compare whether two different non-autonomous systems are qualitatively the same. The equivalence preserves the notion of topological conjugacy when the non-autonomous sytems are actually autonomous.

DEFINITION 1.1. *Suppose X is a topological space. Suppose $\{f_1, f_2, \dots\}$ is a sequence of continuous functions where $f_i : X \longrightarrow X$ for each i. Then $(X, \{f_1, f_2, \dots\})$ is a non-autonomous dynamical system.*

To provide an example, recall from differential equations that a non-autonomous systems is of the form $\frac{dx}{dt} = F(x,t)$. One can construct a discrete non-autonomous dynamical system from this differential equation. First choose a sequence of times $\{t_0, t_1, t_2, t_3, \dots\}$. Then choose an initial point and time (y, t_0) for the differential equation; this choice corresponds to choosing a point $y \in X$. Suppose $\phi_t(y)$ represents the solution to $\frac{dx}{dt} = F(x,t)$ where the initial condition is (y, t_0). Define $f_1(y) = \phi_{t_1}(y)$. Let $x \in X$. To define $f_2(x)$, suppose $\psi_t(x)$ represents the solution to $\frac{dx}{dt} = F(x,t)$, where the initial condition is (x, t_1). Define $f_2(x) = \psi_{t_2}(x)$. To define f_n, proceed with the previous methods inductively. Sometimes we write $(X, \{f_i\})$ to represent the discrete non-autonomous system $(X, \{f_1, f_2, \dots\})$.

A non-autonomous dynamical system starts with a point in X, say x_0, (x_0 is the initial condition) and applies the sequence of functions to the point x_0 in the same order that they are ordered as a sequence. This is analogous to choosing an initial condition (x_0, t_0) for $\frac{dx}{dt} = F(x,t)$, and recording where this point flows to in X at times $\{t_1, t_2, t_3, \dots\}$. This is the notion of the orbit of x_0.

Definition 1.2. *The orbit of the point x_0 with respect to the non-autonomous system $(X, \{f_1, f_2, \dots\})$ is $\{x_0, f_1(x_0), f_2 \circ f_1(x_0), f_3 \circ f_2 \circ f_1(x_0), f_4 \circ f_3 \circ f_2 \circ f_1(x_0), \dots, f_k \circ \dots \circ f_2 \circ f_1(x_0), \dots\}$.*

The next definition qualitatively measures whether the two systems are equivalent. It is a natural extension of topological conjugacy, which qualitatively measures equivalent behavior between two autonomous systems.

Definition 1.3. *Suppose X and Y are topological spaces. The non-autonomous* *systems* $(X, \{f_1, f_2, \dots\})$ *and* $(Y, \{g_1, g_2, \dots\})$ *are dynamically equivalent if there exists a homemorphism* $h : X \longrightarrow Y$ *such that* $h \circ f_k \circ f_{k-1} \cdots f_2 \circ f_1 = g_k \circ g_{k-1} \cdots g_2 \circ g_1 \circ h$ *for all* k.

The function h is what we call the dynamical equivalence. Notice that when $f_k = f$ and $g_k = g$ for all k, then the two dynamical systems are autonomous, and the dynamical equivalence h becomes a topological conjugacy. Topological conjugacy means that there is a homeomorphism $h : X \longrightarrow Y$ that satisfies $h \circ f = g \circ h$.

Suppose there is a sequence of times $\{t_1, t_2, t_3, \dots\}$ satisfying $t_i = t_0 + i\delta$ for some $\delta > 0$. Suppose there is an integer n such that $F(x,t) = F(x, t + n\delta)$ for all x and for all t. If we record the position of (x, t_0) with respect to the differential equation $\frac{dx}{dt} = F(x,t)$ at the times $\{t_1, t_2, t_3, \dots\}$ and define $\{f_1, f_2, f_3, \dots\}$ the same way as above, then after a time elapse of $n\delta$, the sequence of functions $\{f_1, f_2, f_3, \dots\}$ repeats.

DEFINITION 1.4. *Suppose* X *is a topological space. The non-autonomous system* $(X, \{f_1, f_2, \dots\})$ *is periodic with period* n *if the sequence of functions is* $\{f_1, f_2, \dots f_n, f_1, f_2, \dots f_n, f_1, f_2, \dots f_n, \dots\}$. *We say that it has fundamental period* n *if there is no* k *smaller than* n *so that the sequence is* $\{f_1, f_2, \dots f_k, f_1, f_2, \dots f_k, \dots\}$. *When the word fundamental is omitted the context will make it clear that we mean fundamental period.*

We now develop some notation to enable us to refer to the function after k iterates. Define $[g,f]^k : X \longrightarrow X$ as follows. Let $x \in X$. The expression $[g,f]^k(x)$

means that we first apply f to x, and then apply g to $f(x)$ so we have $g(f(x))$. Then apply f so we now have $f \circ g \circ f(x)$. The object is to alternate between applying f and then g, for k times; this is the value of $[g, f]^k(x)$. In other words, apply the sequence $\{f, g, f, g, \dots\}$ k times to x. The expression $[f_p, f_{p-1}, \dots, f_2, f_1]^k(x)$ means that we first apply f_1 to x, and then apply f_2 to $f_1(x)$ to obtain $f_2 \circ f_1(x)$. Then apply f_3. If $k > p$, then after applying f_p, start all over again and apply f_1, f_2, and so on.

Notice that an autonomous dynamical system is a non-autonomous system with period 1. The next issue is to determine when a dynamical equivalence exists between two non-autonomous systems. To start we use a weaker notion of comparison between two non-autonomous systems. In the case when both non-autonomous dynamical systems are periodic, the weaker notion implies dynamical equivalence.

LEMMA 1.1. *Suppose* $(X, \{f_1, \dots, f_n\})$ *and* $(Y, \{g_1, \dots, g_n\})$ *are non-autonomous systems with period* n. *Then* $\{f_1, f_2, \dots, f_n, f_1, f_2, \dots, f_n, f_1, f_2, \dots\}$ *is dynamically equivalent to* $\{g_1, g_2, \dots, g_n, g_1, g_2, \dots, g_n, g_1, g_2, \dots\}$ *if and only if there exists a homeomorphism* $h : X \longrightarrow Y$ *so that the following holds:*

For each r *satisfying* $1 \leq r \leq n$

$$h \circ (f_r \circ f_{r-1} \circ \dots \circ f_1) = (g_r \circ \dots g_2 \circ g_1) \circ h \tag{1.1}$$

Proof: The "only if" part follows immediately from the definition of dynamically equivalent. For the "if" section, it suffices to show that for any natural number

m, we have $h \circ (f_m \circ f_{m-1} \circ \ldots \circ f_1) = (g_m \circ \ldots g_2 \circ g_1) \circ h$. This is done by induction on m. The base case holds by setting $m = 1$.

Using the inductive hypothesis, suppose for all $k < m$ we have $h \circ (f_k \circ f_{k-1} \circ \ldots \circ f_1) = (g_k \circ \ldots g_2 \circ g_1) \circ h$. Then $m = qn + r$ where $0 \leq r < n$, and $h \circ (f_{m+1} \circ f_m \circ \ldots \circ f_1) = h \circ (f_{m+1} \circ \ldots \circ f_{qn+1} \circ f_{qn} \circ \ldots \circ f_1) = h \circ (f_{j+1} \circ \ldots \circ f_1 \circ f_{qn} \circ \ldots \circ f_1) = g_{j+1} \circ \ldots \circ g_1 \circ (h \circ f_{qn} \circ \ldots \circ f_1)$ by setting $j = r + 1$ in the hypothesis. By the induction hypothesis,

$g_{j+1} \circ \ldots \circ g_1 \circ (h \circ f_{qn} \circ \ldots \circ f_1) = g_{j+1} \circ \ldots \circ g_1 \circ g_{qn} \circ \ldots \circ g_1 \circ h = (g_{m+1} \circ \ldots \circ g_1) \circ h$ by the definition of the g_i sequence. ■

Now we turn our attention to a stronger condition than dynamical equivalence. Suppose there exists a homeomorphism $h : X \longrightarrow Y$ so that $h \circ f_i = g_i \circ h$ for all i satisfying $1 \leq i \leq n$. This implies that equation 1.1 above holds for any r. As an example, consider $h \circ f_1 = g_1 \circ h$ by setting $r = 1$. Then $h \circ (f_{r+1} \circ \ldots \circ f_1) = g_{r+1} \circ h \circ (f_r \circ \ldots \circ f_1)$. Setting $i = r + 1$, and applying the inductive hypothesis, we obtain $g_{r+1} \circ h \circ (f_r \circ \ldots \circ f_1) = (g_{r+1} \circ \ldots \circ g_1) \circ h$. However, equation 1.1 holding for all r between 0 and n does not necessarily imply that there is one topological conjugacy h so that $h \circ f_i = g_i \circ h$ for every i. This condition is stronger than dynamical equivalence.

PROPOSITION 1.1. *Suppose there exists a homeomorphism $h : X \longrightarrow Y$ so that $h \circ f_i = g_i \circ h$ for all i. Then h is a dynamical equivalence between the non-autonomous system $\{f_1, f_2, f_3, \ldots\}$ and the system $\{g_1, g_2, g_3, \ldots\}$.*

Proof: By hypothesis, we have the base case $h \circ f_1 = g_1 \circ h$. Using the inductive hypothesis, suppose that $h \circ (f_r \circ \ldots \circ f_1) = (g_r \circ \ldots \circ g_1) \circ h$. Then $h \circ f_{r+1} \circ (f_r \circ \ldots \circ f_1) = g_{r+1} \circ h \circ (f_r \circ \ldots \circ f_1)$ because $h \circ f_i = g_i \circ h$ for all i. The inductive hypothesis allows us to substitute $(g_r \circ \ldots \circ g_1) \circ h$ for $h \circ (f_r \circ \ldots \circ f_1)$, so $h \circ f_{r+1} \circ (f_r \circ \ldots \circ f_1) = g_{r+1} \circ (g_r \circ \ldots \circ g_1) \circ h$. ■

If, however, we do not require h to be the same for every i, the results change drastically. If f_i is topologically conjugate to g_i for every i, are f_i and g_i necessarily dynamically equivalent? The answer is no. Define $f_i, g_i : \mathbb{R} \longrightarrow \mathbb{R}$ as $f_i(x) = \frac{1}{2}x + \frac{1}{2}$ for every i, and $g_1(x) = \frac{1}{2}x + \frac{1}{2}$. Set $g_2(x) = \frac{3}{4}x + \frac{1}{4}$. Inductively, set $g_n(x) = (1 - 2^{-n})x + 2^{-n}$. Note that f_i is topologically conjugate to g_i for every i.

We show that there can not be a dynamical equivalence by contradiction. Suppose there exists a dynamical equivalence h between $(\mathbb{R}, \{f_i\})$, and $(\mathbb{R}, \{g_i\})$. First, notice that $\lim_{k\to\infty} f_k \circ f_{k-1} \ldots f_1(.5) = \lim_{k\to\infty} f^k(.5) = 1$. Since $h \circ f_1(1) = g_1 \circ h(1)$, this implies $h(1) = .9h(1) + .1$. Thus, $h(1) = 1$.

The orbit of $h(.5)$ converges to

$\lim_{k\to\infty} g_k \circ g_{k-1} \ldots g_1 \circ h(.5) = \lim_{k\to\infty} h \circ f_k \ldots f_1(.5) = h(\lim_{k\to\infty} f_k \circ f_{k-1} \ldots f_1(.5)) = 1$. Since h is a homeomorphism, $a = h(.5) \neq 1$. From Example 1.1 (below) we see that $\lim_{k\to\infty} g_k \circ g_{k-1} \ldots g_1(a) \neq 1$ since $a \neq 1$. Thus, $\lim_{n\to\infty} g_n \circ g_{n-1} \circ \ldots g_2 \circ g_1 \circ h(.5) \neq \lim_{n\to\infty} h \circ f_n \circ f_{n-1} \circ \ldots f_2 \circ f_1(.5)$. This is a contradiction, so a dynamical equivalence h can not exist.

EXAMPLE 1.1. *We construct a sequence of real-valued functions $\{g_1, g_2, \ldots\}$ so that each g_i considered as its own autonomous systems has a stable fixed point at $x = 1$. However, the fixed point $x = 1$ is not stable with respect to the non-autonomous system $\{g_1, g_2, \ldots\}$ in the following sense: $\lim_{k\to\infty} g_k \circ g_{k-1} \ldots g_1(x) < 1$ for any $x < 1$,* *and* $\lim_{n\to\infty} g_k \circ g_{k-1} \dots g_1(x) > 1$ *for any* $x > 1$.

We first require a definition of infinite products and a theorem about infinite products [APOSTOL].

DEFINITION 1.5. *Given an infinite product* $\prod_{n=1}^{\infty} u_n$, *let* $p_m = \prod_{n=1}^{m} u_n$. *If no factor* u_n *is zero, we say the product converges if there exists a number* $p \neq 0$ *such that the sequence* $\{p_m\}$ *converges to* p. *In this case,* p *is called the value of the product and we write* $p = \prod_{n=1}^{\infty} u_n$. *If* $\{p_m\}$ *converges to zero, we say the product diverges to zero.*

THEOREM 1.1. *Assume that* $a_n \geq 0$. *Then the product* $\prod_{n=1}^{\infty}(1 - a_n)$ *converges if and only if the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof: [APOSTOL].

We observe that $g_n(x) = s_n x + t_n$ where $s_n = (1 - 2^{-n})x$ and $t_n = 2^{-n}$. We proceed with the first few iterates, $g_2 \circ g_1(x) = s_2(s_1 x + t_1) + t_2$, and
$g_3 \circ g_2 \circ g_1(x) = s_3[s_2(s_1 x + t_1) + t_2] + t_3 = s_3 s_2 s_1 x + s_3 s_2 t_1 + s_3 t_2 + t_3$.
By induction,

$$g_k \circ g_{k-1} \dots g_1(x) = (\prod_{i=1}^{k} s_i)x + ((\prod_{i=2}^{k} s_i)t_1 + (\prod_{i=3}^{k} s_i)t_2 + \dots s_k t_{k-1} + t_k.$$

Hence, $\lim_{n\to\infty} g_n \circ g_{n-1} \dots g_1(x) = (\prod_{i=1}^{\infty} s_i)x + (\prod_{i=2}^{\infty} s_i)t_1 + (\prod_{i=3}^{\infty} s_i)t_2 + \cdots =$
$(\prod_{i=1}^{\infty} s_i)x + \sum_{k=2}^{\infty}(\prod_{i=k}^{\infty} s_i)t_{k-1}$. Notice $g_n(1) = 1$ for any n, and $|g_n{}'(1)| = s_n < 1$, so 1 is a stable fixed point for each i. Thus, for any k, we obtain $\lim_{k\to\infty} g_k \circ g_{k-1} \dots g_1(1) =$
$(\prod_{i=1}^{\infty} s_i) + \sum_{k=2}^{\infty}(\prod_{i=k}^{\infty} s_i)t_{k-1} = 1$. Hence, if $x < 1$, then $(\prod_{i=1}^{\infty} s_i)x + \sum_{k=2}^{\infty}(\prod_{i=k}^{\infty} s_i)t_{k-1} <$

$(\prod_{i=1}^{\infty} s_i) + \sum_{k=2}^{\infty} (\prod_{i=k}^{\infty} s_i) t_{k-1} = 1$. This inequality is strict because $(\prod_{i=1}^{\infty} s_i) > 0$ follows from Theorem 1.1. When $x < 1$, we obtain $\lim_{n \to \infty} g_n \circ g_{n-1} \dots g_1(x) < 1$. Since the functions g_i are symmetrical about the diagonal $g(x) = x$, when $x > 1$, we have $\lim_{n \to \infty} g_n \circ g_{n-1} \dots g_1(x) > 1$. ■

Now we turn our attention to definitions about the behavior of the orbit of a point with respect to a fixed non-autonomous system. We show that dynamical equivalence preserves certain properties of an orbit. This means that dynamical equivalence is a useful way of judging when two dynamical systems are qualitatively the same.

DEFINITION 1.6. *A point $p \in X$ is a periodic point of the non-autonomous system $(X, \{f_i\})$ with period k if for all $m \in \mathbb{N}$, $f_{mk} \circ f_{mk-1} \circ \dots f_2 \circ f_1(p) = p$.*

Notice that this notion of periodic point has to do with the orbit of a point in X, while the notion of periodic for a non-autonomous system has to do with the periodicity of the functions applied. We now show that dynamical equivalences preserve periodic orbits.

REMARK 1.1. *Suppose the non-autonomous systems $(X, \{f_1, f_2, \dots\})$ and $(Y, \{g_1, g_2 \dots\})$ are dynamically equivalent. Let $h : X \longrightarrow Y$ be a dynamical equivalence. If p has period k with respect to $(X, \{f_1, f_2, \dots\})$, then $h(p)$ has period k with respect to $(Y, \{g_1, g_2, \dots\})$.*

Proof: For each natural number n, we have $h(p) = h \circ f_{nk} \circ f_{nk-1} \circ \ldots f_2 \circ f_1(p) g_{nk} \circ g_{nk-1} \circ \ldots g_2 \circ g_1(h(p))$. ■

This next definition captures the intuitive notion that even though an orbit may not be periodic, it still may return arbitrarily close to its initial starting point an infinite number of times. The definition reduces to the standard definition of non-wandering point for autonomous dynamical systems.

DEFINITION 1.7. *A point $x \in X$ is a non-wandering point of the non-autonomous system $(X, \{f_i\})$ if for each neighborhood U of x there exists $k > 0$, (k is dependent on x and U) and a $q \in U$ so that $f_k \circ f_{k-1} \ldots f_2 \circ f_1(q) \in U$. Let $\Omega(\{f_i\})$ denote the set of all non-wandering points with respect to the non-autonomous system $\{f_i\}$.*

REMARK 1.2. *Any periodic point is a non-wandering point.*

LEMMA 1.2. *A dynamical equivalence maps non-wandering points to non-wandering points. Formally, if h is a dynamical equivalence between $(X, \{f_i\})$, and $(Y, \{g_i\})$, then p is a non-wandering point of $(X, \{f_i\})$ if and only if $h(p)$ is a non-wandering point of $(Y, \{g_i\})$.*

Proof: Let p be a non-wandering point of f_i. Consider $h(p) \in Y$. Let U be a neighborhood of $h(p)$. Then there exists $k > 0$, and a $q \in h^{-1}(U)$ so that $f_k \circ f_{k-1} \ldots f_1(q) \in h^{-1}(U)$. But this implies that $h \circ f_k \circ f_{k-1} \ldots f_1(q) \in U$. In turn, because h is a dynamical equivalence, we have that $g_k \circ g_{k-1} \cdots g_1 \circ h(q) \in U$. Also, $h(q) \in U$, so $h(p)$ is a non-wandering point for g_i. Since h is a homeomorphism the reverse argument holds. From this, we conclude that h maps the non-wandering points of f_i homeomorphically onto the non-wandering points of g_i. ■

This Lemma is important because in Section IV, we show that the topological entropy of a non-autonomous system on X equals the topological entropy restricted to the non-wandering points. The next Remark originates from standard results about non-wandering points in autonomous systems [BOWEN].

REMARK 1.3. *The set of non-wandering points of any non-autonomous dynamical system* $(X, \{f_1, f_2, f_3, \dots\})$ *is a closed set in* X.

Proof: Suppose $\{x_n\}$ is a sequence of non-wandering points and $x_n \to p$. (We argue by contradiction.) Suppose p is not a non-wandering point. Then there exists an open neighborhood U of p such that $U \cap \bigcup_{n=1}^{\infty} f_n \circ f_{n-1} \circ \dots f_2 \circ f_1(U) = \emptyset$. Since $x_n \to p$, there is a large enough m such that $x_m \in U$. Since U is open, there exists an open set W satisfying $x_m \in W \subset U$. This means that $W \cap \bigcup_{n=1}^{\infty} f_n \circ f_{n-1} \circ \dots f_2 \circ f_1(W) \subset U \cap \bigcup_{n=1}^{\infty} f_n \circ f_{n-1} \circ \dots f_2 \circ f_1(U) = \emptyset$. Since W is an open set containing x_m, this contradicts that x_m is a non-wandering point. ■

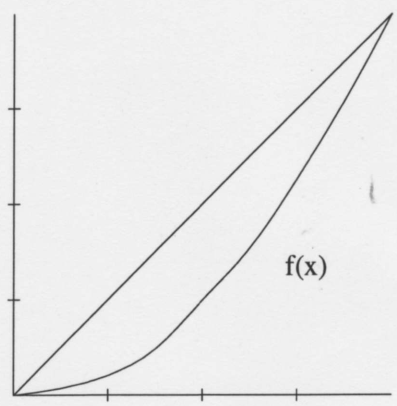
Again, following the standard results about non-wandering points for autonomous systems [BOWEN], we have the definition of an invariant set.

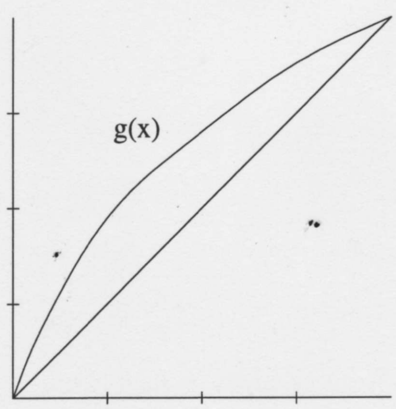
DEFINITION 1.8. *Let* V *be a subset of* X. *Let* $(X, \{f_i\})$ *be a non-autonomous sys-*

tem. The set V *is* $\{f_i\}$ *invariant if for all* k *we have that* $f_k \circ f_{k-1} \circ f_2 \circ f_1(V) \subset V$

REMARK 1.4. *The set of non-wandering points is not* $\{f_i\}$ *invariant. The set is not even* f_i *invariant for a period* 2 *non-autonomous system i.e.* $\{f, g, f, g, f, g, \ldots\}$

The following example verifies Remark 1.4. Define $f, g : [0,1] \longrightarrow [0,1]$ as follows. Set $f(x) = x^2$ when $\frac{1}{2} \leq x \leq 1$ and $4x^4$ when $0 \leq x \leq \frac{1}{2}$. Set $g(x) = \sqrt{x}$. The function sequence is $\{f, g, f, g, f, g, \ldots\}$.





Claim: All points in the open interval $(0, \frac{1}{2})$ are wandering points.

Let $p \in (0, \frac{1}{2})$. Then $f(p) = p^2 < \frac{1}{4}$. If $\epsilon < \frac{1}{2} - p$, then $g \circ f(p+\epsilon) = 2(p+\epsilon)^2 = 2p^2 + \epsilon(4p+\epsilon)$. Because $2p^2 = (2p)p < p$, we may choose $\epsilon > 0$ small enough so that $\epsilon(4p+\epsilon) < p - 2p^2$. Thus, $2(p+\epsilon)^2 < p$. Set $\delta = \min\{(p - p^2), (p - 2(p+\epsilon)^2)\}$. Define the open interval $U = (p - \delta, p + \epsilon)$. From the previous calculations, we see that $f(U) = ((p-\delta)^2, (p+\epsilon)^2)$ and $(p+\epsilon)^2 < 2(p+\epsilon)^2 \leq p - \delta$.

Thus, $f(U) \cap U = \emptyset$. Further, $g \circ f(U) \subset [0, 2(p+\epsilon)^2] \subset [0, p-\delta]$ because $2(p+\epsilon)^2 \leq p - \delta$. Hence, $g \circ f(U) \cap U = \emptyset$. Now for any $x < \frac{1}{2}$, we have $g \circ f(x) < x$ and $f(x) < x$. Thus, for any $x \in (g \circ f)(U)$, we have $(g \circ f)^k(x) \leq p - \delta$ for all $k \in \mathbb{N}$, and we have $f \circ (g \circ f)^k(x) \leq p - \delta$ for all $k \in \mathbb{N}$. Thus, $(g \circ f)^k(U) \cap U = \emptyset$, and $f \circ (g \circ f)^k(U) \cap U = \emptyset$ for all $k \in \mathbb{N}$. $\blacksquare$.

Notice that $f(0) = 0$, so $\{0\}$ is a non-wandering point. If $p \geq \frac{1}{2}$, then $f(p) = p^2$ and $g \circ f(p) = \sqrt{p^2} = p$. Hence, the set of non-wandering points of the non-autonomous system $\{f, g, f, g, \ldots\}$ is the set $[\frac{1}{2}, 1] \cup \{0\}$.

Now the goal is to find a non-wandering point c satisfying $f(c) = p$ such that p lies in $(0, \frac{1}{2})$. Set $c = \sqrt{\frac{2}{5}}$. Thus, $c > \sqrt{\frac{1}{4}} = \frac{1}{2}$, so $g \circ f(c) = g(c^2) = c$. Hence, c is a non-wandering point. Further, $p = f(c) = \frac{2}{5}$. Thus, p is a wandering point. We conclude that the non-wandering points are not f_i invariant, even for a period two non-autonomous system. $\blacksquare$.